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Continuous Numerical Method of Collocation Interpolation for the Solution of Wave Equations Using Laguerre Polynomial approximation

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Abstract
A new Continuous interpolant method based on polynomial approximation is here proposed for solving wave equation subject to some initial and boundary conditions. The method results from discretization of the wave equation which leads to the production of a system of algebraic equations. By solving the system of algebraic equations by employing the continuous interpolant scheme we obtain the problem approximate solutions.

Keywords: Polynomials, interpolation, collocation, wave equation, lines, Continuous interpolant

1. Introduction

There is a growing interest in the recent literatures concerning continuous numerical methods for solving ODEs. In science and engineering, this interest is extended to the development of continuous numerical techniques for solving wave equation subject to initial and boundary conditions. Their advantages over discrete ones are now well known, including their connection to large families (Odekunle, 2008). We presented an extension of this continuous method for solving ODEs to solve PDEs in two dimensions as a conjecture. Hitherto, efforts have been on top gear to derive continuous numerical interpolant for solving wave equation. When this is achieved then a generalized scheme that can solve all the branches of PDEs- parabolic, hyperbolic and elliptic equations is possible. In this paper therefore, we develop a new continuous numerical interpolant which is based on interpolation and collocation at some points along the coordinates (Adam & David 2002).

2. Solution Method

In 2002, Awoyemi postulated that in setting up the solution method we select an integer N such that $N > 0$. Then subdivide the interval $0 \le x \le X$ into N equal subintervals with mesh points along the space coordinate given by $x_i = ih,\ i = \frac{1}{\beta}\left(\frac{1}{\beta}\right)N$, where $Nh = X,\ \beta \neq 0$. Similarly, reverse the roles of x and t select another integer M such that $M > 0$. Also, subdivide the interval $0 \le t \le T$ into M equal subintervals with mesh points along time axis given by $t_j = jk$, $j = \frac{1}{\alpha}\left(\frac{1}{\alpha}\right)M$ where $Mk = T,\ \alpha \neq 0$ and h, k are the mesh sizes along space and time axes respectively(Odekunle,2008; Biazar& Ebrahimi,2005). Here, we seek for the approximate solutions $\overline{U}(x,t)$ to $\overline{U}_{p-1}(x,t)$ in the manner of Yildiz (2001) and Zheyin (2012) of the form

$$\overline{U}(x,t) \approx \overline{U}_{p-1}(x,t) = \sum_{r=0}^{p-1} a_r \left[\phi_r(x,t)\right],\ x \in [x_i, x_{i+h}], \tag{2.0}$$

Over $h > 0, k > 0$ mesh sizes, such that

$$0 = x_0 < \ldots < x_1 < \ldots x_N,\ 0 = t_0 < \ldots < t_1 < \ldots t_M.$$

Bao et al., (2003) suggested that in doing this we let ρ be the sum of interpolation points along space and time coordinates. Therefore, $\rho = g + b$, where g is the number of interpolation points along the space axis and b the number of interpolation points along time coordinate. The basis function $\phi_r(x,t),\ r = 0,1,\ldots,p-1$ is the Laguerre's polynomials which is known, a_r are the constants to be determined. There will be flexibility in the choice of the basis function as may be desired for specific application. For this work, we consider the Laguerre's polynomial $\phi_r(x,t) = x^r t^r$. The interpolation values $\overline{U}_{i,j},\ldots,\overline{U}_{i+h-1,j}$ are assumed to have been determined from previous steps, while the method seeks to obtain $\overline{U}_{i+h,j}$ (Odekunle, 2008; Awoyemi, 2002; Benner and Mena, 2004; Dehghan,2003). Applying the above interpolation conditions on eqn. (2.0) we obtain;

$$a_0\phi_0(x_{i+h}, t_{j+k}) + a_1\phi_1(x_{i+h}, t_{j+k}) + .. a_{p-2}\phi_{p-1}(x_{i+h}, t_{j+k}) = \overline{U}(x_{i+h}, t_{j+k}) \tag{2.1}$$

We let $h = -\frac{1}{\beta}\left(\frac{1}{\beta}\right)\left[g - \left(\frac{2\beta - 1}{\beta}\right)\right]$ arbitrarily and $k = 0$, then by Crammer's rule, eqn. (2.1) becomes

$$\left.\begin{array}{l} W\underline{a} = \underline{F}, \quad \underline{F} = \left(\overline{U}_{V,j}, \overline{U}_{v+\frac{1}{\beta},j}, \ldots, U_{z,j}\right)^T \\ \underline{a} = \left(a_0, \ldots, a_{p-1}\right)^T \end{array}\right\}$$

(2.2)

and

$$W = \begin{bmatrix} \phi_0(x_v, t_j), & \phi_1(x_v, t_j), & \ldots, & \phi_{p-1}(x_v, t_j \\ \phi_0\left(x_{v+\frac{1}{\beta}}, t_j\right), & \phi_1\left(x_{v+\frac{1}{\beta}}, t_j\right), & \ldots, & \phi_{p-1}\left(x_{v+\frac{1}{\beta}}\right. \\ \ldots, & \ldots, & \ldots, & \\ \phi_0(x_z, t_j), & \phi_1(x_z, t_j), & \ldots, & \phi_{p-1}(x_z, t_j \end{bmatrix}$$

Where $z = i + g - \left(\frac{2\beta - 1}{\beta}\right)$, $v = i - \frac{1}{\beta}$ and W^{-1} exists,and again $\alpha, \beta \neq 0$ (Odekunle, 2008; Bensoussan, et al., 2007). Hence, by equation (2.2) we obtain

$$\underline{a} = \overline{\omega}\underline{F}, \quad \overline{\omega} = W^{-1}$$

(2.3)

The vector $\underline{a} = \left(a_0, \ldots, a_{p-1}\right)^T$ is now determined in terms of known parameters in $\overline{\omega}\underline{F}$. If $\overline{\omega}_{r+1}$ is the $(r+1)^{th}$ row of $\overline{\omega}$ then

$$a_r = \overline{\omega}_{r+1}\underline{F}$$

(2.4)

Eqn. (2.4) determines the values of a_r. Let us take first and second derivatives of eqn. (2.0) with respect to x,

$$\overline{U}'(x,t) = \sum_{r=0}^{p-1} a_r\left[\phi_r'(x,t)\right]$$

$$\overline{U}''(x,t) = \sum_{r=0}^{p-1} a_r\left[\phi_r''(x,t)\right]$$

(2.5)

Substituting eqn. (2.4) into eqn. (2.5), we obtain

$$\overline{U}''(x,t) = \sum_{r=0}^{p-1}\left[\overline{\omega}_{r+1}\underline{F}\left(\phi_r''(x,t)\right)\right]$$

(2.6)

Again, by Odekunle (2006) we reverse the roles of x and t in eqn. (2.1) and we arbitrarily set $k = 0\left(\frac{1}{\alpha}\right)\left[b - \left(\frac{\alpha - 1}{\alpha}\right)\right]$ and $k = 0, \ \alpha \neq 0$, then again by Cramer's rule eqn. (2.1) becomes.

$$\left.\begin{array}{l} Y\underline{a} = \underline{E}, \quad \underline{E} = \left(\overline{U}_{i,\eta-\frac{1}{\alpha}}, \overline{U}_{i,\eta}, \ldots, U_{i,\gamma}\right)^T \\ \underline{a} = \left(a_0, \ldots, a_{p-1}\right)^T \end{array}\right\}$$

(2.7)

and

$$Y = \begin{bmatrix} \phi_0\left(x_i, t_{\eta-\frac{1}{\alpha}}\right), & \phi_1\left(x_i, t_{\eta-\frac{1}{\alpha}}\right), & \dots, & \phi_{p-1}\left(x \right. \\ \phi_0(x_i, t_\eta), & \phi_1(x_i, t_\eta), & \dots, & \phi_{p-1}(x_i \\ \dots, & \dots, & \dots, & \\ \phi_0(x_i, t_\gamma), & \phi_1(x_i, t_\lambda), & \dots, & \phi_{p-1}(x_i \end{bmatrix}$$

Where $\eta = j + \frac{1}{\alpha}, \quad \gamma = j + b - \left(\frac{\alpha - 1}{\alpha}\right),$

and Y^{-1} exists and $\alpha \neq 0$ (Eyaya, 2010; Penzl, 2000; Pierre, 2008). Hence from equation (2.7) we obtain

$$\underline{a} = L\underline{E}, \quad L = Y^{-1}$$

(2.8)

The vector $\underline{a} = (a_0, \dots, a_{p-1})^T$ is now determined in terms of known parameters in $L\underline{E}$. If L_{r+1} is the $(r+1)^{th}$ row of L then

$$a_r = L_{r+1}\underline{E}$$

(2.9)

Also, eqn. (2.9) determines the values of a_r. Taking the first and second derivatives of eqn. (2.0) with respect to t, we obtain

$$\overline{U}'(x,t) = \sum_{r=0}^{p-1} a_r \left[\phi_r'(x,t)\right]$$

$$\overline{U}''(x,t) = \sum_{r=0}^{p-1} a_r \left[\phi_r''(x,t)\right] \quad (2.10)$$

Substituting eqn. (2.9) in eqn. (2.10) we have

$$\overline{U}''(x,t) = \sum_{r=0}^{p-1} \left[L_{r+1}\underline{E}\left(\phi_r''(x,t)\right)\right]$$

(2.11)

But by eqn. (1.0) it is obvious that eqn. (2.11) is equal to eqn. (2.6), therefore,

$$\sum_{r=0}^{p-1} \left[L_{r+1}\underline{E}\left(\phi_r''(x,t)\right)\right] - \sum_{r=0}^{p-1} \left[\overline{\varpi}_{r+1}\underline{F}\left(\phi_r''(x,t)\right)\right] = 0$$

(2.12)

Collocating eqn. (2.12) at $x = x_i$ and $t = t_j$ we obtain a new continuous numerical interpolant that solves eqn. (1.0) explicitly.

2.1 Numerical Examples

In this section we give some numerical examples to compute approximate solutions for equation (1.0) by the method discussed in this paper. This is in order to test the numerical accuracy of the new method. To achieve this, we follow Richard et al., (2001) and Saumaya et al., (2012),we truncate the Laguerre's polynomial after second order and use it as the basis function for the computation. The resultant interpolant is used to solve the following test problems.

Example 3.0

Use the scheme to approximate the solution to the wave equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0, \quad 0 < x < 1 \qquad 0 < t \qquad ,$$

$$U(0,t) = U(1,t) = 0,\ t > 0$$

$$U(x,0) = \sin \pi x, \qquad 0 \le x \le 1, \quad \frac{\partial U}{\partial x}(x,0) = 0, \qquad 0 \le x \le 1$$

Table I: Result of action of Eqn. (2.12) on example 3.0

x	Exact solution $U(x,t)$	Schmidt method $U(x,t)$	New Method $U(x,t)$	Errors	
				New Method	Schmidt method
0	0	0	0	0	0
0.1	0.305212482	0.305992120	0.305235901	2.3419 X E-5	7.7963840 X E- 4
0.2	0.580548640	0.582031600	0.580593187	4.4547 X E-5	1.4829604 X E -3
0.3	0.799056652	0. 801097772	0.799117966	6.1314 X E-5	2.0411200 X E- 3
0.4	0.939347432	0.941746912	0.939419511	7.2079 X E-5	2.3994802 X E -3
0.5	0.987688340	0.990211303	0.987764129	7.5789 X E-5	2.5229632 X E -3
0.6	0.939347432	0.941746912	0.939419511	7.2079 X E-5	2.3994802 X E -3
0.7	0.799056652	0. 801097772	0.799117966	6.1314 X E-5	2.0411200 X E- 3
0.8	0.580548640	0.582031600	0.580593187	4.4547 X E-5	2.0411200 X E- 3
0.9	0.305212482	0.305992120	0.305235901	2.3419 X E-5	7.7963840 X E- 4
1	0	0	0	0	0

Example 3.1

Use the scheme to approximate the solution to the wave equation

$$\frac{\partial^2 U}{\partial t^2} - 4\frac{\partial^2 U}{\partial x^2} = 0 \qquad 0 < x < 1, \qquad 0 < t\,, U(0,t) = U(1,t) = 0\,, \qquad t > 0$$

$$U(x,0) = \sin \pi x, \qquad 0 \le x \le 1,\ \frac{\partial U}{\partial x}(x,0) = 0, \qquad 0 \le x \le 1$$

Table II: Result of action of Eqn. (2.12) on example 3.1

x	Exact Solution $U(x,t)$	Schmidt method $U(x,t)$	New method $U(x,t)$	Errors	
				New Method	Schmidt Method
0	0	0	0	0	0
0.1	0.305212482	0.304983829	0.305235901	2.3419 X E-5	2.2865 X E -4
0.2	0.58054864	0.580113718	0.580593187	4.4547 X E-5	4.3492 X E -4
0.3	0.799056652	0.798458034	0.799117966	6.1314X E-5	5.9862 X E -4
0.4	0.939347432	0.9386437114	0.939419511	7.2079 X E-5	7.0372 X E– 4
0.5	0.987688340	0.986948407	0.987764129	7.5789 X E-5	7.3993 X E– 4
0.6	0.939347432	0.305992120	0.939419511	7.2079 X E-5	7.0372 X E– 4
0.7	0.799056652	0.798458034	0.799117966	6.1314 X E-5	5.9862 X E -4
0.8	0.58054864	0.580113718	0.580593187	4.4547 X E-5	4.3492 X E -4
0.9	0.305212482	0.304983829	0.305235901	2.3419 X E-5	2.2865 X E -4
1	0	0	0	0	0

3. Discussion of Results

Results of action of eqn. (2.12) which is our new continuous interpolant scheme have shown that the new off - grid method is more accurate than the known explicit Schmidt method when used to solve wave equations subject to some initial and boundary conditions. These numerical results have confirmed the validity of this new off -grid continuous interpolant method. Again, this continuous method has also provided a more stable scheme than the known explicit Schmidt method as will be seen in our subsequent research results.

3.1 Recommendations

Based on our findings and restrictions experienced during the derivation of these schemes, we intend to base our recommendations on some salient and could easily be overlooked areas. Our suggestions follow our desire to investigate higher fractional mesh points that could have easily given us more accurate results. Following these arguments, we wish to present the following points as areas that need some investigations and further researching: To make generalization of the new method, proper investigation into the non-variability of the number of collocation points have to be investigated. When this investigation is successful, then we can easily generalize our scheme to solveeven parabolic and elliptic equations as while.

Also, we wish to challenge our teeming researchers to come up with a new scheme with higher fractional mesh points. This we believe will give us a more accurate result. We are again, suggesting that researchers should try to vary the number of collocation points, while keeping the interpolation point fixed.

Conclusion

In this work, an off - grid continuous interpolant is developed for solving wave equationssubject to some initial and boundary conditions. The Laguerre's polynomial is employed as basis function in the derivation of our new scheme. Interpolation and collocation at various points were also carry - out. Complex functional evaluations were avoided through - out the processes of derivation of this method. Besides, the new method had avoided the complex expansion of Laguerre's polynomial as is seen in the derivations of most explicit methods. These had helped in reducing the number of procedures during program writing. Also, the new method has improved the accuracy of the results obtained through the avoidance of error that might be caused by truncation in the approximation of the Laguerre's polynomial.

When compared with the Schmidt scheme the results showed that the performances of the new scheme were far better than the Schmidt scheme as is shown in tables I & II above.

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