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A New Weibull Inverted Exponential Distribution: Properties and Applications

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Abstract
This study introduces a two-parameter model called a new Weibull inverted exponential (AWIE) distribution for modeling lifetime datasets. The proposed model uses the new Weibull-X characterizations. The density of the proposed model can be symmetrical, left skewed, reverse J-shaped, unimodal and increasing, and bathtub shaped failure rate. Some mathematical properties of the proposed distribution are investigated. The two parameters of the proposed model are obtained by maximum likelihood. The regression and Bayesian models of the proposed model are obtained in a class form. Two real-life data sets are used to illustrate the applicability of the proposed model. The results of the test statistics show that the proposed new model gives a better fit compared to some related distributions in literature.

Keywords: Exponential model, Generating function, Inverted exponential, Weibull model

1. Introduction
The quality of any statistical decision made depends on the probability distribution used in the estimation. Hence, considerable statistical research has been made in distribution theory to develop a new class of distribution that is more flexible and relevant.

The inverted exponential distribution is a continuous model with inverted bathtub hazard function introduced in Keller et al. (1982). This inverted exponential distribution became very useful in modeling Poisson processes between events in which the exponential distribution could not handle. This also was as a result of its constant failure rate. Though, the exponential distribution is used to model Poisson processes with memoryless random processes. However, because of its constant failure rate and since life-time scenarios are not constant; the inverted exponential model is introduced to address this inefficiency.

In recent years, the inverted exponential model has been used in medicine, engineering, biology, business, electronic systems and insurance. Unal et al. (2018) proposed the alpha power inverted exponential with application to survival time of patients with neck and head cancer diseases. Oguntunde et al. (2014a) proposed the exponentiated generalized inverted exponential distribution. Oguntunde et al. (2014b) proposed the Kumaraswamy inverse exponential distribution. Abouammoh and Alshingiti (2009) proposed the generalized inverted exponential distribution. Efe-Eye et al. (2020) proposed the Weibull alpha power inverted exponential distribution with application to glass fibers and carbon data. Eghwerido et al. (2019) proposed the extended new generalized exponential distribution. Eghwerido et al. (2020)

proposed the Gompertz alpha power inverted exponential distribution. Zelibe et al. (2019) proposed the Kumaraswamy alpha power inverted exponential distribution. Singh and Goel (2015) proposed beta inverted exponential distribution. Chandrakant et al. (2018) proposed the Weibull inverse exponential Distribution. Oguntunde et al. (2017) proposed a useful extension of the inverse exponential distribution. Alrajhi (2019) proposed the odd Frechet inverse exponential distribution.Oguntunde Adejumo (2015) proposed transmuted inverse exponential distribution. Singh et al. (2013) estimated the parameters of generalized inverted exponential. Oguntunde et al. (2018) proposed the Gompertz inverse exponential distribution. Truncated inverted generalized exponential was proposed in Genc (2015), and exponentiated shifted exponential was proposed in Agu (2020).

The motivation of this article is to introduce a class of inverted exponential distribution with increasing, reversed J-shape and bathtub shaped hazard rate model for real-life data.

This article aims at a two-parameter model called AWIE distribution for modeling Poisson processes between events. The mathematical properties of the AWIE model together with its Bayesian, unit model and regression model were also developed. The major characteristic of the proposed model is that one shape parameter is added to make the new proposed model more flexible.

Let $w > 0$ be the scale parameter, then the probability density function (pdf) and cumulative distribution function (cdf) of the inverted exponential distribution for a random variable U is expressed as

$$f(u) = \frac{w}{u^2}\exp\left(-\frac{w}{u}\right), \quad u > 0, \tag{1}$$

and

$$F(u) = \exp\left(-\frac{w}{u}\right), \quad u > 0. \tag{2}$$

Suppose $f(u)$ and $F(u)$ are the baseline pdf and cdf of a particular distribution. Now, consider a one parameter Weibull cdf $F(w) = 1 - \exp(-u^w)$, for $u \geq 0, w > 0$. Then, the cdf of the new Weibull-X defined in Zubair et al. (2018) is expressed as

$$G(u) = \alpha \int_0^{\frac{-\log(1-F(u))}{1-F(u)}} u^{\alpha-1}\exp(-u^\alpha)du$$
$$= 1 - \exp\left[-\left(\frac{1-F(u)}{-\log(1-F(u))}\right)^{-\alpha}\right], \quad u \in \Re, \alpha > 0. \tag{3}$$

The corresponding density is defined as

$$g(u) = \alpha \frac{f(u)[-\log(1-F(u))]^{\alpha-1}}{[1-F(u)]^{\alpha+1}[1-\log(1-F(u))]^{-1}}$$
$$\times \exp\left[-\left(\frac{1-F(u)}{-\log(1-F(u))}\right)^{-\alpha}\right], \quad u \in \Re, \alpha > 0. \tag{4}$$

2. The AWIE Distribution

Let U be a random variable. Thus, motivated by the new Weibull-X method, the AWIE distribution is introduced with cdf given as

$$G(u)=1-\exp\left[-\left(\frac{1-\exp\left(-\frac{w}{u}\right)}{-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)}\right)^{-\alpha}\right],\quad u\in\Re,\alpha>0 \tag{5}$$

The density that corresponds is defined as

$$g(u)=\frac{\frac{w}{u^2}\exp\left(-\frac{w}{u}\right)\left[-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)\right]^{\alpha-1}\alpha}{\left[1-\exp\left(-\frac{w}{u}\right)\right]^{\alpha+1}\left[1-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)\right]^{-1}}\times\exp\left[-\left(\frac{1-\exp\left(-\frac{w}{u}\right)}{-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)}\right)^{-\alpha}\right], \tag{6}$$

for $w>0, u>0, \alpha\in\Re$ and α is additional parameter for controlling the kurtosis and skewness.

Figure 1 shows the plot of the AWIE density model for different parameter values cases. The plot shows that the density can be decreasing, increasing, unimodal and skewed.

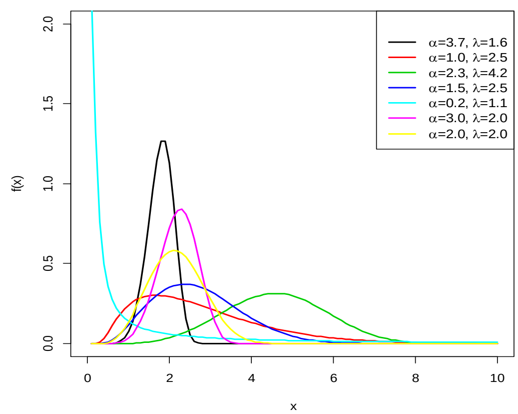

Figure 1. *The AWIE density for different parameter values cases*

However, omitting the dependence of the parameter w and α, we can simply write $g(u)=g(u;w,\alpha)$ and $G(u)=G(u;w,\alpha)$. The reliability function that corresponds to Equation (6) is given as

$$S(u)=\exp\left[-\left(\frac{1-\exp\left(-\frac{w}{u}\right)}{-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)}\right)^{-\alpha}\right],\quad u\in\Re,\alpha>0 \tag{7}$$

The failure rate is expressed as

$$h(u)=\frac{\frac{w}{u^2}\exp\left(-\frac{w}{u}\right)\left[-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)\right]^{\alpha-1}\alpha}{\left[1-\exp\left(-\frac{w}{u}\right)\right]^{\alpha+1}\left[1-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)\right]^{-1}},\quad u\in\Re,\alpha>0 \tag{8}$$

Figure 2 shows the plot of the AWIE failure rate model for different parameter values cases. The plot shows that the failure rate can be bathtub, increasing and J-shaped.

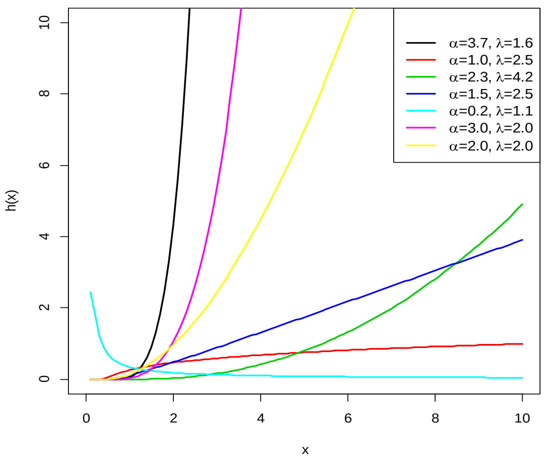

Figure 2. *The AWIE hazard rate function for different parameter values cases*

The AWIE cumulative hazard rate (H), reversed hazard rate (r) and odds (O) functions are expressed as

$$H(u)=-\left[-\left(\frac{1-\exp\left(-\frac{w}{u}\right)}{-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)}\right)^{-\alpha}\right],\quad u\in\Re,\alpha>0 \tag{9}$$

$$r(u)=\left[\begin{array}{l}\alpha\dfrac{\frac{w}{u^{2}}\exp\left(-\frac{w}{u}\right)\left[-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)\right]^{\alpha-1}}{\left[1-\exp\left(-\frac{w}{u}\right)\right]^{\alpha+1}\left[1-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)\right]^{-1}}\\ \times\exp\left[-\left(\dfrac{1-\exp\left(-\frac{w}{u}\right)}{-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)}\right)^{-\alpha}\right]\end{array}\right]$$
$$\times\left[1-\exp\left[-\left(\frac{1-\exp\left(-\frac{w}{u}\right)}{-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)}\right)^{-\alpha}\right]\right]^{-1},\quad u\in\Re,\alpha>0, \tag{10}$$

$$O(u)=\left[1-\exp\left[-\left(\frac{1-\exp\left(-\frac{w}{u}\right)}{-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)}\right)^{-\alpha}\right]\right]$$
$$\times\left[\exp\left[-\left(\frac{1-\exp\left(-\frac{w}{u}\right)}{-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)}\right)^{-\alpha}\right]\right]^{-1}. \tag{11}$$

3. Linear Representation

This section discusses the linear representation of the AWIE model. This mixture mathematical representation derived is used to obtain some simplified properties of the proposed model. Thus, it helps to express the new model in terms of the inverted exponential distribution.

Lemma 2.1. For any real value $[-\log(1-A)]^{\beta}$

$$= A^{\beta} + \sum_{m=0}^{\infty} D_m(\beta) A^{m+\beta+1}$$

where $D_0(\beta) = \frac{1}{2}\beta,$

$$D_1(\beta) = \frac{1}{24}[\beta(3\beta+5)],$$

$$D_2(\beta) = \frac{1}{48}[\beta(\beta^2+5\beta+6)],$$

$$D_3(\beta) = \frac{1}{5760}[15\beta^3+150\beta^2+485\beta+502]$$

are the Stirling polynomials.

Lemma 2.2. For any real value a,

$$\exp(au) = \sum_{p=0}^{\infty} \frac{a^p u^p}{p!} .$$

Lemma 2.3. For any real value c,

$$(1-\mathrm{A})^{-c} = \sum_{h=0}^{\infty} \frac{\Gamma(c+h)}{h!\Gamma(c)} A^h,$$

provided $|A|<1,\ c>0$ *and* $\Gamma(\cdot)$ *is the gamma function.*

However, by Lemma 3. 1, 3.2 and 3.3, the cdf of the AWIE model can be expressed as

$$G(u) = 1 - \sum_{h,p=0}^{\infty} (-1)^p \frac{1}{h!\,p!} \frac{\Gamma(p\alpha+h)}{\Gamma(p\alpha)} \times \exp\left(-\frac{w}{u}(h+p\alpha)\right) B \qquad (12)$$

where

$$B = \left[1 + \sum_{m=0}^{\infty} D_m(p\alpha) \exp\left(-\frac{w}{u}(m+1)\right)\right].$$

Also, the pdf is given as

$$g(u) = \sum_{h,p=0}^{\infty} Avio_{h,p} \frac{w}{u^2} \exp\left(-\frac{w}{u}(\alpha(p+1)(h+2)-h)\right) S \qquad (13)$$

where

$$S = \left[1 + \sum_{m=0}^{\infty} D_m(\alpha(p+1)) \exp\left(-\frac{w}{u}(m+1)\right) - 1\right] \times \left[\exp\left(\frac{w}{u}\right) + \sum_{m=0}^{\infty} D_m(-1) \exp\left(-\frac{mw}{u}\right) - 1\right]$$

B and S converge to a unit as $u \to \infty$.

4. Statistical Properties

In this section, the statistical properties of the proposed AWIE distribution are derived and investigated. This include the moments, moment generating function, probability generating function and order statistics.

4.1 Moments

Lemma 3.1. For $m \in Z$, where Z is a positive integer, then

$$\int_l^k u^{m-2} \exp\left(-\frac{w}{u}\right) du = (-1)^m \frac{w^{m-1}}{(m-1)!}\left[E_i\left(-\frac{w}{k}\right) - E_i\left(-\frac{w}{l}\right)\right] + \sum_{j=1}^{m-1} \frac{C_{m-j}(-w)^{j-1}}{(m-1)(m-2)(m-3)\ldots(m-j)}$$

where

$C_{m-j} = \exp\left(-\frac{w}{k}\right)k^{m-j} - \exp\left(-\frac{w}{u}\right)l^{m-j}$, and

$E_i(n) = -\int_n^{\infty} \frac{1}{z}\exp(-z)dz,\ n>0$ is the exponential integral function.

Thus, the r^{th} moment of U for the AWIE model is expressed

$$\mu_r' = \sum_{h,p=0}^{\infty} Avio_{h,p} \left[\begin{array}{l} (-1)^m \dfrac{(w(\alpha(p+1)(h+2)-h))^{m-1}}{(m-1)!} \\ \times \left[\begin{array}{l} E_i\left(-\dfrac{w(\alpha(p+1)(h+2)-h)}{k}\right) \\ -E_i\left(-\dfrac{w(\alpha(p+1)(h+2)-h)}{l}\right) \end{array} \right] \\ +\sum_{j=1}^{m-1} \dfrac{C_{m-j}(-w(\alpha(p+1)(h+2)-h))^{j-1}}{(m-1)(m-2)(m-3)\ldots(m-j)} \end{array} \right]. \quad (14)$$

where

$$C_{m-j} = \exp\left(-\frac{w(\alpha(p+1)(h+2)-h)}{k}\right)k^{m-j} - \exp\left(-\frac{w(\alpha(p+1)(h+2)-h)}{u}\right)l^{m-j}$$

4.2 The AWIE Probability Weighted Moments (PWM)

The PWM of the AWIE distribution of the random variable U is expressed as

$$M_{r,s}(u) = \sum_{h,p=0}^{\infty} wAvio_{h,p} \int_0^{\infty} u^{r-2^s} \times \exp\left(-\frac{w}{u}(\alpha(p+1)(h+2)-h)\right) \times \left[1-\sum_{h,p=0}^{\infty}\exp\left(-\frac{w}{u}(h+p\alpha)\right)\right] du \quad (15)$$

Lemma 3.1. for $c > 0,\ (1-A)^c = \sum_{k=0}^{c}(-1)^{c-k}\binom{c}{k}A^k$.

Thus, the last quantity in Equation (15) is expressed as

$$\left[1-\sum_{h,p=0}^{\infty}\exp\left(-\frac{w}{u}(h+p\alpha)\right)\right]^s = \sum_{h,p=0}^{\infty}\sum_{k=0}^{c}(-1)^{c-k}\binom{s}{k}\exp\left(-\frac{sw}{u}(h+p\alpha)\right).$$

Hence,

$$M_{r,s}(u) = \sum_{h,p=0}^{\infty}\sum_{k=0}^{c} wAvio_{h,p} \int_0^{\infty} u^{r-2}\exp\left(-\frac{w}{u}(h(s-1)+sp\alpha+\alpha(p+1)(h+2))\right)du$$

$$= \sum_{h,p=0}^{\infty}\sum_{k=0}^{c} \left[\begin{array}{l} (-1)^m \dfrac{(w(h(s-1)+sp\alpha+\alpha(p+1)(h+2)))^{m-1}}{(m-1)!} \\ \times \left[\begin{array}{l} E_i\left(-\dfrac{w(h(s-1)+sp\alpha+\alpha(p+1)(h+2))}{k}\right) \\ -E_i\left(-\dfrac{w(h(s-1)+sp\alpha+\alpha(p+1)(h+2))}{l}\right) \end{array} \right] \\ +\sum_{j=1}^{m-1} \dfrac{C_{m-j}(-w(h(s-1)+sp\alpha+\alpha(p+1)(h+2)))^{j-1}}{(m-1)(m-2)(m-3)\ldots(m-j)} \end{array} \right] \quad (16)$$

4.3 The AWIE Generating Function

The probability generating function for the AWIE distribution and the moment generating functions are obtained in this section.

The probability generating function of the AWIE random variable U is obtained as

$$V(t)=\sum_{b=0}^{\infty}\frac{(\log t)^{b}}{t!}\int_{0}^{\infty}u^{b}g(u)du \quad . \tag{17}$$

After integrating and simplifying, we have

$$V(t)=\sum_{b=0}^{\infty}\sum_{h,p=0}^{\infty}\frac{Avio_{h,p}(\log t)^{b}}{t!}\times\left[\begin{array}{l}(-1)^{b}\dfrac{(w(\alpha(p+1)(h+2)-h))^{b-1}}{(b-1)!}\\ \times\left[\begin{array}{l}E_{i}\left(-\dfrac{w(\alpha(p+1)(h+2)-h)}{k}\right)\\ -E_{i}\left(-\dfrac{w(\alpha(p+1)(h+2)-h)}{l}\right)\end{array}\right]\\ +\sum_{j=1}^{b-1}\dfrac{C_{b-j}(-w(\alpha(p+1)(h+2)-h))^{j-1}}{(b-1)(b-2)(b-3)\ldots(b-j)}\end{array}\right] \tag{18}$$

The moment generating function is defined

$$M(t)=\sum_{h,p=0}^{\infty}\int_{-\infty}^{\infty}\exp(tu)g(u)du. \tag{19}$$

4.4 The AWIE Order Statistics

The k^{th} order statistics of the random variable $u_1, u \;\ldots u_n$ that are AWIE distributed is given as

$$g_k(u)=\frac{\alpha n!}{(n-k)!(k-1)!}\left[1-\exp\left[-\left(\frac{1-F(u)}{-\log(1-F(u))}\right)^{-\alpha}\right]\right]^{k-1}\left[\exp\left[-\left(\frac{1-F(u)}{-\log(1-F(u))}\right)^{-\alpha}\right]\right]^{n-k}$$
$$\times\frac{\frac{w}{u^{2}}\exp\left(-\frac{w}{u}\right)\left[-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)\right]^{\alpha-1}}{\left[1-\exp\left(-\frac{w}{u}\right)\right]^{\alpha+1}\left[1-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)\right]^{-1}}\times\exp\left[-\left(\frac{1-\exp\left(-\frac{w}{u}\right)}{-\log\left(1-\exp\left(-\frac{w}{u}\right)\right)}\right)^{-\alpha}\right] \tag{20}$$

4.5 Parameter Estimation

In this section, the maximum likelihood method is employed to obtain the parameters of the AWIE distribution. Let $u=(u_1,u_2,\ldots u_n)$ be the AWIE random sample with unknown parameter vector $K=(w,\alpha)^{T}$. Then, the log-likelihood function ℓ for Equation (6) where,

$d = 1 - \exp\left(-\frac{w}{u}\right),\ L = [-\log(d)], Y = \left[\frac{L}{d}\right]^{\alpha},$ and $P = [1 + L]$ is given as

$$\ell = n\log\alpha + n\log w - \sum_{i=1}^{n} u_i^2 - \sum_{i=1}^{n}\frac{w}{u_i} + (\alpha-1)\sum_{i=1}^{n}\log L_i - (\alpha+1)\sum_{i=1}^{n}\log d_i + \sum_{i=1}^{n} P_i - \sum_{i=1}^{n}[Y_i] \quad (21)$$

Thus, the partial derivatives of Equation (20) with respect to the unknown parameters and equating to zero are expressed as

$$\frac{\partial\ell}{\partial\alpha} = n\alpha^{-1} + \log\sum_{i=1}^{n} L_i - \log\sum_{i=1}^{n} d_i - \sum_{i=1}^{n} Y'_{i,\alpha} = 0, \quad (22)$$

$$\frac{\partial\ell}{\partial w} = nw^{-1} - \sum_{i=1}^{n} u_i^{-1} + (\alpha-1)\sum_{i=1}^{n}\frac{L'_{i,w}}{L_i} - (\alpha+1)\sum_{i=1}^{n}\frac{d'_{i,w}}{d_i} + \sum_{i=1}^{n}\frac{P'_{i,w}}{P_i} - \sum_{i=1}^{n} Y'_{i,w} = 0 \quad (23)$$

The parameter estimates of the unknown can be obtained numerically by Newton-Raphson algorithm in MATHEMATICAL, R, MATLAB and MAPLE.

4.6 The quantile function

Lemma 4.1. For $-1 \le u < 1,\ \log(1-u) = -\sum_{r=1}^{\infty}\frac{u^r}{r!}$

Thus, for U an AWIE random variable and by Lemma 2.2 and Lemma 4.1. Then, the quantile function for $v \in (0,1)$ can be defined in power series as

$$u_v = \left[\frac{\sum_{i=0,r=1}^{\infty}\frac{(-1)^{2i} w^{2i} r^i}{r!2i!} + (-\log(1-v))^{\frac{1}{\alpha}}}{(-\log(1-v))^{\frac{1}{\alpha}}}\right]^{\frac{1}{i}}. \quad (24)$$

when $v = \frac{1}{2}$ in Equation(23), we obtained the median Q_2 as

$$Q_2 = \left[\frac{\sum_{i=0}^{\infty}\sum_{r=1}^{\infty}\frac{(-1)^i w^i r^i}{r!i!} + (-\log(0.5))^{\frac{1}{\alpha}}\sum_{i=0}^{\infty}\frac{(-1)^i w^i}{i!}}{(-\log(0.5))^{\frac{1}{\alpha}}}\right]^{\frac{1}{i}}. \quad (25)$$

However, the first quantile Q_1 and third quantile Q_3 can be obtained respectively as

$$Q_1 = \left[\frac{\sum_{i=0}^{\infty}\sum_{r=1}^{\infty}\frac{(-1)^i w^i r^i}{r!i!} + (-\log(0.75))^{\frac{1}{\alpha}}\sum_{i=0}^{\infty}\frac{(-1)^i w^i}{i!}}{(-\log(0.75))^{\frac{1}{\alpha}}}\right]^{\frac{1}{i}}, \quad (26)$$

and

$$Q_3 = \left[\frac{\sum_{i=0}^{\infty}\sum_{r=1}^{\infty}\frac{(-1)^i w^i r^i}{r!i!} + (-\log(0.25))^{\frac{1}{\alpha}}\sum_{i=0}^{\infty}\frac{(-1)^i w^i}{i!}}{(-\log(0.25))^{\frac{1}{\alpha}}}\right]^{\frac{1}{i}}. \quad (27)$$

4.7 The Simulation Study with the AWIE Distribution

A Monte Carlo simulation study is performed to examine the performance, applicability and flexibility of the AWIE model. The simulation is performed as follows

- Random data are generated using the AWIE quantile function defined as .

$$u_v = \left[\frac{\sum_{i=0,r=1}^{\infty} \frac{(-1)^{2i} w^{2i} r^i}{r!2i!} + (-\log(1-v))^{\frac{1}{\alpha}}}{(-\log(1-v))^{\frac{1}{\alpha}}} \right]^{\frac{1}{i}}$$

- The parameter values are set as follow $\alpha = 0.5$ and $w = 0.6$.
- The random sample sizes are taken as follows: 10, 50,100 and 200.
- Each of the random sample size is replicated 50000 times.

The simulation investigated the mean estimates (AEs), biases, variances and mean squared errors (MSEs) of the AWIE model maximum likelihood estimates.

The maximum likelihood estimates (MLE) is estimated as

$\hat{M}SE_M = \frac{1}{50000} \sum_{i=1}^{50000} \left(\hat{M}_i - M \right)^2$. The bias is obtained as $\hat{B}ias_M = \frac{1}{50000} \sum_{i=1}^{50000} \left(\hat{M}_i - M \right)$.

Table 1 shows the Monte Carlo simulation results. The Table 1 shows that the MSE, variance and biases of the parameter estimates decreases as the sample sizes increases.

Table 1. *Monte Carlo simulation results for mean estimates, estimated biases, variance and mean squared errors*

n	Parameter	AE	Variance	Bias	MSE
10	$\alpha = 0.5$	0.0164	0.0019	-0.4836	0.1058
	$w = 0.6$	2.1956	0.3049	1.5956	1.1423
50	$\alpha = 0.5$	0.0040	0.0001	-0.4960	0.0461
	$w = 0.6$	1.7609	0.2402	1.0619	0.0891
100	$\alpha = 0.5$	0.0037	0.0001	-0.4970	0.0060
	$w = 0.6$	1.5006	0.1205	0.9356	0.0120
200	$\alpha = 0.5$	0.0015	0.0000	-0.5955	0.0026
	$w = 0.6$	1.0070	0.1085	0.0930	0.0037

5. Applications

In this section, the application to real-life data is investigated to enhance the applicability of the proposed model. The goodness-of-fit statistics of the AWIE distribution is compared to the Kumaraswamy alpha power exponential (KAPIE) distribution, Exponential (E), alpha power shifted exponential (APSE) distribution, Inverted exponential (IE) distribution Keller et al. (1982), Alpha power inverted exponential (APIE) distribution [2], generalized inverted generalized exponential (GIGE) distribution (Oguntunde and Adejumo 2015), Weibull Frechet distribution (Afify et al. 2016) and Kumaraswamy Inverted Exponential (KIE). The goodness-of-fit are based on test statistics of Alkaike Information Criteria (AIC), Hanniquin Information Criteria (HQIC), Bayesian Information Criteria

(BIC), Consistent Akaike Information Criteria (CAIC), Cramer-von Mises (W), Anderson Darling (A), and the p values (p-val).

The first data consist of 1.5 cm strengths of glass fibres data at the UK National Physical Laboratory as used in (Smith and Naylor 1987, Bourguignon et al. (2014), Haq 2016, Merovci et al. (2016), Rastogi and Oguntunde(2018), Obubu et al. (2019), Eghwerido et al. (2019), Zelibe et al. (2019), Efe-Eyefia et al. (2020), Eghwerido et al. (2020a), Eghwerido et al. (2020), Eghweridoand Agu (2021), Nzei et al. (2020),Eghwerido et al. (2020), Eghwerido et al. (2020d), Eghwerido et al. (2020b)and Agu and Eghwerido (2021). Tables 2 and 3 provide the results of the different model test statistics for the data

Table 2. *Results of test statistics for fitted models to glass fiber data with standard errors (in parentheses)*

Models	MLEs	AIC	CAIC	BIC	HQIC
AWIE	$\hat{\alpha}=3.1601(0.3170)$	37.2261	37.4261	41.5124	38.9119
	$\hat{w}=1.3554(0.0316)$				
WFr	$\hat{\alpha}=0.3923(0.8023)$	38.79601	39.48567	47.36855	42.16763
	$\hat{w}=0.2476(0.2956)$				
	$\hat{\beta}=1.4897(4.7849)$				
	$\hat{a}=16.8619(20.4912)$				
GIGE	$\hat{\alpha}=163.1990(88.7584)$	50.1241	50.5308	56.5535	52.6528
	$\hat{w}=3.1976(0.0001)$				
	$\hat{\lambda}=2.5489(0.0\ 001$				
APSE	$\hat{\alpha}=8.282858(0.0001)$	50.8399	51.2466	57.2693	53.3686
	$\hat{w}=2.6850(0.2295)$				
	$\hat{\lambda}=1.0530(0.0237)$				
KAPIE	$\hat{\alpha}=1.0442(0.3224)$	52.71052	53.40017	61.28306	56.08214
	$\hat{w}=19.3039(10.5912)$				
	$\hat{\beta}=7.4276(2.0082)$				

	$\hat{\lambda} = 0.0020(0.0007)$				
KIE	$\hat{\alpha} = 3.0231(42.6757)$	53.42339	53.83017	59.85279	55.95211
	$\hat{w} = 163.2151(88.7824)$				
	$\hat{\lambda} = 2.6961(38.0593)$				
E	$\hat{\lambda} = 0.6637(18.9319)$	179.6028	181.8201	185.9439	179.7028
IE	$\hat{\alpha} = 1.4083(0.1774)$	180.8784	180.944	183.0215	181.7213
APIE	$\hat{\alpha} = 83.4506(79.284)$	196.3351	196.5390	200.6167	198.0108
	$\hat{w} = 0.3195(0.0854)$				

Table 3. *Results of Cramer-von Mises (W), Anderson Darling (A), and the p values (p-val). test statistics for fitted models to glass fiber data*

Models	W	A	p-val
AWIE	0.2186	1.2316	0.0960
WFr	0.2471	1.3565	0.0709
GIGE	0.4813	2.6324	0.0134
APSE	0.00001	0.0023	0.0183
KAPIE	0.5063777	2.7706	0.0076
KIE	0.5113	2.8324	0.0000
E	0.9969	4.2902	0.0034
IE	0.9375	5.0649	0.0000
APIE	0.7775	4.2384	0.0000

The results of the test statistics in Table 2 show that the AWIE model has the lowest AIC value. Hence, is consider as best fit for this data set when compared to other two, three and four parameters models such as the Kumaraswamy alpha power exponential (KAPIE) distribution [9], exponential (E), alpha power shifted exponential (APSE) distribution [19], inverted exponential (IE) distribution [1], alpha power inverted exponential (APIE) distribution [2], generalized inverted generalized exponential (GIGE) distribution [20], Weibull Frechet distribution [21] and Kumaraswamy inverted exponential (KIE)

6. Conclusion

This study proposes a new class of exponential distribution named AWIE distribution. The new model extends the exponential distribution for analysing real-life data. Some of its statistical structural properties were examined. The AWIE distribution was also expressed as a linear function of the exponential distribution. The model parameters were obtained by maximum likelihood method. A simulation study was used to illustrate the performance of the proposed model. A real-life application was further used to investigate the efficiency of the proposed model. It was observed that the proposed model provide a better fit compared to some existing statistical models like Kumaraswamy alpha power exponential (KAPIE) distribution, Exponential (E) distribution, alpha power shifted exponential (APSE) distribution, inverted exponential (IE) distribution, alpha power inverted exponential (APIE) distribution, generalized inverted generalized exponential (GIGE) distribution, Weibull Frechet distribution and Kumaraswamy inverted exponential (KIE). The proposed model can also be applied to the field of survival lifetime data, economics, hydrology, growth rate in modeling and others.

Conflicts of interest

The authors declared that there is no conflict of interest.

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