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Approximate Solutions of N-Dimensional Klein-Gordon Equation with Yukawa Potential: Application to Diatomic molecules

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Abstract

We solved the Klein-Gordon equation with Yukawa potential usingthe Nikiforov-Uvarov method. In order to overcome the centrifugal barrier, the Greene and Aldrich approximation scheme was employed. The energy eigenvalues for relativistic, non- relativistic and the corresponding normalized wave function were obtained. A special case of Coulomb potential was obtained. The energy eigenvalues equation were used to study some selected diatomic molecules such as N_2, CO, NO, and CH.The bound state energy eigenvalues expressions and numerical computations agreed with the existing literature.

Keywords: Schrödinger equation; Nikiforov-Uvarov method, Klein-Gordon equation; Diatomic molecules

1. Introduction

The bound state solutions of the Klein – Gordon (KG) equation are only possible for some potentials of physical interest (Inyang*et al*., 2021; Allosh*et al*., 2021). These solutions could be exact or approximate and they contain all the necessary information to describe the quantum system (Abu-shady*et al*.,2021;Ikot*et al*., 2020). The exact solutions of the Schrödinger equation (SE) for a hydrogen atom and for a harmonic oscillator represent two typical examples in quantum mechanics(Mutuk,2018).

The solutions of the SE with different potential models have been investigated by many authors (Rani*et al*., 2018;Ciftci and Kisoglu,2018;Oyewumiand Oluwadare,2016). Also, different methods have been employed in obtaining either exact or approximate solutions of the SE or KG such as, Laplace transformation method (Abu-Shady and Khokha, 2018; Abu-Shady*et al*.,2018),super symmetric quantum mechanics (SUSYQM) (Abu-Shady and Ikot, 2019;Al-Jamel, 2019;Onate*et al*.,2021), the Nikiforov-Uvarov (NU) method (Ntibi*et al*.,2020;Okoi*et al*.,2020;Edet*et al*.,2019;Inyang*et al*.,2021;.Edet et al.,2020; Inyang*et al*., 2020; Inyang *et al*., 2021;Edet*et al*., 2020; Ekpo *et al*., 2020; William*et al*., 2020;Inyang*et al*.,2021;Okon*et al*., 2016; Abu-Shady *et al*., 2019;Inyang *et al*., 2021; Omugbe,2020; Thompson *et al*.,2022; Abu-Shady, 2016;Akpan *et al*., 2021;Inyang*et al*.,2021; Inyang *et al*., 2021), the Nikiforov-Uvarov Functional Analysis (NUFA) method (Ikot *et al*., 2021; Rampho*et al*.,2020), Series expansion method (SEM) (Inyang *et al*., 2020;Ibekwe *et al*., 2020; Inyang *et al*., 2021; Abu-Shady,and Fath-Allah, 2019;Inyang*et al*.,2021;Ibekwe *et al*.,2021), analytical exact iterative method(AEIM)(Khokha*et al*.,2016),WKB approximation method (Omugbe*et al*., 2020; Omugbe*et al*., 2021;Omugbe *et al*., 2022; Omugbe,2020; Omugbe, 2020;Hitler *et al*.,2017) ,Exact Quantization Rule (EQR) (Qiang *et al*., 2008;Inyang *et al*.,2020)and others(Ali *et al*., 2020).

The Yukawa potential (YP) is an effective non-relativistic potential describing the strong interactions between nucleons. It takes the form;

$$V(r) = -\frac{A_0 e^{-\alpha r}}{r} \tag{1}$$

And can be seen as a standard version of the Coulomb potential if $\alpha = 0$,with A_0 describing the strength of the potential and α is the screening parameter as shown in Fig. 1.

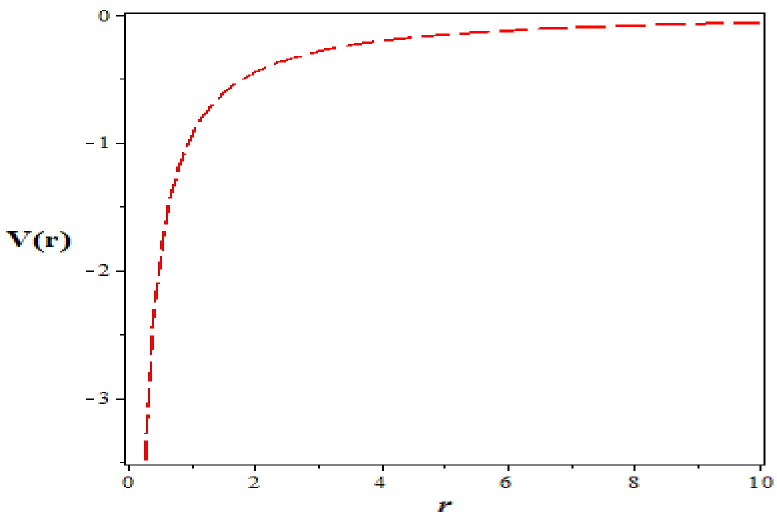

Figure 1: Plots of Yukawa potential with r in (fm^{-1})

This potential is often used to compute bound-state normalizations and energy levels of natural atomswhich have been studied over the past years(Edet*et al*., 2020;Ekwevugbe,2020; Okoi*et al*.,2020; Edet *et al*.,2020;Nwabuzor *et al*., 2021;Onate *et al*.,2021; Ikot *et al*.,2019). The KG equation containing a four-vector linear momentum operator and a rest mass requires introducing the four-vector potential $V(r)$ and a space time scalar potential $S(r)$. With the configuration $S(r) = V(r)$ or $S(r) = -V(r)$, it has been shown extensively in literature that the KG equation share the same energy spectrum.While for $S(r) = V(r) = 2V(r)$, gives non-relativistic limits of the equation conforming exactly to that of the SE.The YP, has been used extensively by many authors in obtaining the energy of the bound state in atomic, nuclear, and particle physics (Horchani et al., 2021;Purohit *et al*., 2021).For instance,Chakrabarti and Das 2016 presented a perturbative solution of the Riccati equation leading to an analytic superpotential for Yukawa potential. Ikhdair and Sever 2006, investigated energy levels ofneutral atoms by applying an alternative perturbative scheme in solving the SE for the YP model. In this present study, we are motivated by the current trend in the study of bound state problems, to investigate theapproximate bound state solutions of theKlein-Gordon equation with Yukawa potential and apply the energy eigenvalues equation to study some selected diatomic molecules.

Therefore, to obtain approximate solutions, we employ a suitable approximation scheme. It is found that such approximation proposed by Greene and Aldrich,1956.

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\left(1 - e^{-\alpha r}\right)^2} \quad (2)$$

Is a good approximation to the centrifugal term which is valid for α <<1 for a short-rangepotential.

1.1. Review of Nikiforov-Uvarov(NU) method

The NU method was proposed by Nikiforov and Uvarov to transform Schrödinger-like equations into a second-order differential equation via a coordinate transformation $s = s(r)$, of the form.

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{3}$$

where $\tilde{\sigma}$(s), and $\sigma(s)$ are polynomials, at most second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. The exact solution of Eq. (3) can be obtained by using the transformation.

$$\psi(s)=\phi(s)y(s) \tag{4}$$

This transformation reduces Eq.(3) into a hypergeometric-type equation of the form

$$\sigma(s)y''(s)+\tau(s)y'(s)+\lambda y(s)=0 \tag{5}$$

The function $\Phi(s)$ can be defined as the logarithm derivative

$$\frac{\phi'(s)}{\phi(s)}=\frac{\pi(s)}{\sigma(s)} \tag{6}$$

with $\pi(s)$ being at most a first-degree polynomial. The second part of $\psi(s)$ being y(s) in Eq.(5) is the hypergeometric function with its polynomial solution given by Rodrigues relation as

$$y(s)=\frac{B_{nl}(s)}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right] \tag{7}$$

where B_n is the normalization constant and $\rho(s)$ the weight function which satisfies the condition below;

$$\left(\sigma(s)\rho(s)\right)'=\tau(s)\rho(s) \tag{8}$$

where also

$$\tau(s)=\tilde{\tau}(s)+2\pi(s) \tag{9}$$

For bound solutions, it is required that

$$\frac{d\tau(s)}{ds}<0 \tag{10}$$

The eigenfunction and eigenvalues can be obtained using the definition of the following function $\pi(s)$ and parameter λ, respectively:

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)} \tag{11}$$

and

$$\lambda=k_-+\pi'_-(s) \tag{12}$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (11) equal to zero.As such, the new eigenvalues equation can be given as

$$\lambda + n\tau^{'}(s) + \frac{n(n-1)}{2}\sigma^{"}(s) = 0, (n = 0,1,2,...) \quad (13)$$

2. Approximate solutions of the Klein-Gordon equation with Yukawa potential

The Kelein-Gordon equation for a spinless particle for $\hbar = c = 1$ in D-dimensions is given as

$$\left[-\nabla^2 + \left(M + S(r)\right)^2 + \frac{\left(N+2l-1\right)\left(N+2l-3\right)}{4r^2}\right]\psi\left(r,\theta,\varphi\right) = \left[E - V(r)\right]^2\psi\left(r,\theta,\varphi\right)$$
(14)

where ∇^2 is the Laplacian, M is the reduced mass, E is the energy spectrum and n and l are the radial and orbital angular momentum quantum numbers respectively or vibration-rotation quantum number in quantum chemistry. It is a common practice that for the wavefunction to satisfy the boundary conditions it can be rewritten as

$$\psi\left(r,\theta,\varphi\right) = \frac{R_{nl}}{r}Y_{lm}\left(\theta,\varphi\right) \quad (15)$$

The angular component of the wavefunction could be separated leaving only the radial part as shown below

$$\frac{d^2R(r)}{dr^2} + \left[\left(E^2 - M^2\right) + V^2(r) - S^2(r) - 2\left(EV(r) + MS(r)\right) - \frac{\left(N+2l-1\right)\left(N+2l-3\right)}{4r^2}\right]R(r) = 0 \quad (16)$$

Thus, for equal vector and scalar potentials $V(r) = S(r) = 2V(r)$ then Eq.(16) becomes

$$\frac{d^2R(r)}{dr^2} + \left[\left(E^2 - M^2\right) - 2V(r)\left(E + M\right) - \frac{\left(N+2l-1\right)\left(N+2l-3\right)}{4r^2}\right]R(r) = 0$$
(17)

We substitute Eq.(1) into Eq.(17) and obtain

$$\frac{d^2R(r)}{dr^2} + \left[\left(E^2 - M^2\right) + \frac{A_0e^{-\alpha r}}{r}\left(E_{nl} + M\right) - \frac{\left(N+2l-1\right)\left(N+2l-3\right)}{4r^2}\right]R(r) = 0$$
(18)

We transform the coordinate of Eq.(18) by setting

$$s = e^{-\alpha r} \quad (19)$$

Differentiating Eq.(19) and simplifying gives;

$$\frac{d^2R(\mathrm{r})}{dr^2} = \alpha^2 s^2\frac{d^2R(s)}{ds^2} + \alpha^2 s\frac{dR(s)}{ds} \quad (20)$$

Substituting Eqs.(2),(19) and (20) into Eq.(18) gives,

$$\frac{d^2R(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dR(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[-\varepsilon(1-s)^2 + \beta\left(s - s^2\right) - \gamma\right]R(s) = 0,$$
(21)

where

$$\left.\begin{aligned} -\varepsilon &= \frac{E_{nl}^2 - M^2}{\alpha^2} \\ \beta &= \frac{A_0\left(E_{nl} + M\right)}{\alpha} \\ \gamma &= \frac{\left(N+2l-1\right)\left(N+2l-3\right)}{4} \end{aligned}\right\} \quad (22)$$

Expanding the square bracket of Eq.(21) we obtain

$$\frac{d^2R(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR(s)}{ds}+\frac{1}{s^2(1-s)^2}\left[-(\varepsilon+\beta)s^2+(2\varepsilon+\beta)s-(\varepsilon+\gamma)\right]R(s)=0 \quad (23)$$

Comparing Eq.(23) with Eq.(3) we obtain the following parameters

$$\left.\begin{aligned}\tilde{\tau}(s)&=1-s\\ \sigma(s)&=s(1-s)\\ \sigma'(s)&=1-2s\\ \tilde{\sigma}(s)&=-(\varepsilon+\beta)s^2+(2\varepsilon+\beta)s-(\varepsilon+\gamma)\end{aligned}\right\} \quad (24)$$

Substituting Eq.(24) into Eq.(11) we have

$$\pi(s)=-\frac{s}{2}\pm\sqrt{(A-k)s^2+(B+k)s+C} \quad (25)$$

where

$$\left.A=\frac{1}{4}+\varepsilon+\beta,\ B=-(2\varepsilon+\beta),\ C=\varepsilon+\gamma\right\} \quad (26)$$

To find the constant, k , the discriminant of the expression under the square root of Eq.(25) must be equal to zero. As such, we have that

$$k=-(B+2C)-2\sqrt{C}\sqrt{C+B+A} \quad (27)$$

Substituting Eq.(26) into Eq.(27) we have

$$k_-=\beta-2\gamma-2\sqrt{\varepsilon+\gamma}\sqrt{\frac{1}{4}+\gamma} \quad (28)$$

Substituting Eq.(27) into Eq.(25) we have

$$\pi(s)=-\frac{s}{2}\pm\left[\left(\sqrt{C}+\sqrt{C+B+A}\right)s-\sqrt{C}\right] \quad (29)$$

Substituting Eq.(26) into Eq.(29) we have

$$\pi_-(s)=-\frac{s}{2}-\left[\left(\sqrt{\varepsilon+\gamma}+\sqrt{\frac{1}{4}+\gamma}\right)s-\sqrt{\varepsilon+\gamma}\right] \quad (30)$$

Differentiating Eq.(30) we have

$$\pi'_-(s)=-\frac{1}{2}-\left(\sqrt{\varepsilon+\gamma}+\sqrt{\frac{1}{4}+\gamma}\right) \quad (31)$$

By substituting Eqs. (28) and (31)into Eq.(12) gives,

$$\lambda=\beta-2\gamma-2\sqrt{\varepsilon+\gamma}\sqrt{\frac{1}{4}+\gamma}-\frac{1}{2}-\left(\sqrt{\varepsilon+\gamma}+\sqrt{\frac{1}{4}+\gamma}\right) \quad (32)$$

With $\tau(s)$ being obtained from Eq.(9) as

$$\tau(s)=1-2s+2\sqrt{\varepsilon+\gamma}s-2\sqrt{\frac{1}{4}+\gamma}s+2\sqrt{\varepsilon+\gamma} \quad (33)$$

Differentiating Eq.(33) yields

$$\tau'(s)=-2-2\left(\sqrt{\varepsilon+\gamma}+\sqrt{\frac{1}{4}+\gamma}\right) \quad (34)$$

And also taking the second derivative of $\sigma'(s)$ with respect to s from Eq.(24),we have

$$\sigma''(s)=-2 \quad (35)$$

Substituting Eqs.(34) and (35) into Eq.(13) and simplifying, yields

$$\lambda_n = n^2 + n + 2n\sqrt{\varepsilon+\gamma} + 2n\sqrt{\frac{1}{4}+\gamma} \quad (36)$$

Equating Eqs.(32) and (36) and substituting Eq.(22) yields the energy eigenvalue equation of the Yukawa potential in the relativistic limit as

$$M^2 - E^2 = -\alpha^2\left(\frac{(N+2l-1)(N+2l-3)}{4}\right) + \frac{\alpha^2}{4}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(N+2l-1)(N+2l-3)}{4}}\right)^2 - \frac{A_0(E_{nl}+M)}{\alpha} + \frac{(N+2l-1)(N+2l-3)}{4}}{n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(N+2l-1)(N+2l-3)}{4}}}\right]^2 \quad (37)$$

2.1 Non-Relativistic Limit

In this section, we consider the non-relativistic limit of Eq.(37). Considering a transformation of the form: $M+E \to \frac{2\mu}{\hbar^2}$ and $M-E \to -E$ and substitute it into Eq.(37), we have the non relativistic energy eigenvalue equation as

$$E_{nl} = \frac{\hbar^2\alpha^2}{2\mu}\left(\frac{(N+2l-1)(N+2l-3)}{4}\right) - \frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(N+2l-1)(N+2l-3)}{4}}\right)^2 - \frac{2\mu A_0}{\alpha\hbar^2} + \frac{(N+2l-1)(N+2l-3)}{4}}{n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{(N+2l-1)(N+2l-3)}{4}}}\right]^2 \quad (38)$$

We take a special case of Eq.(1) by setting $\alpha = 0$ and $N = 3$ to obtain the energy eigenvalues of Coulomb potential as

$$E_{nl} = -\frac{\mu A_0^2}{2\hbar^2(n+l+1)^2}$$

(38a)

To obtain the corresponding wavefunction, we consider Eq.(6) and upon substituting Eqs.(24) and (33) and integrating, we get

$$\phi(\mathrm{s}) = s^{\sqrt{\varepsilon+\gamma}}(1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma}}$$

(39)

To get the hypergeometric function considering Eq.(4) ,we first determine the weight function

of Eq.(8) , upon differentiating the left hand side we obtain

$$\frac{\rho'(s)}{\rho}=\frac{\tau(s)-\sigma'(s)}{\sigma(s)}$$
(40)

Substituting Eqs.(24) and (33) into Eq.(40) and integrating ,thereafter simplify we obtain

$$\rho(s)=s^{2\sqrt{\varepsilon+\gamma}}\left(1-s\right)^{2\sqrt{\frac{1}{4}+\gamma}}$$
(41)

By substituting Eqs.(24) and (41) into Eq.(7) we obtainthe Rodrigue's equation as

$$y_n(s)=B_{nl}s^{-2\sqrt{\varepsilon+\gamma}}\left(1-s\right)^{-2\sqrt{\frac{1}{4}+\gamma}}\frac{d^n}{ds^n}\left[s^{n+2\sqrt{\varepsilon+\gamma}}\left(1-s\right)^{n+2\sqrt{\frac{1}{4}+\gamma}}\right] \quad (42)$$

where B_{nl}= normalization constant.

Equation (42) is a equivalent to

$$P_n^{\left(2\sqrt{\varepsilon+\gamma},2\sqrt{\frac{1}{4}+\gamma}\right)}\left(1-2s\right)$$
(43)

where P_n= Jacobi Polynomial

The wave function is given by

$$\psi_{nl}(s)=B_{nl}s^{\sqrt{\varepsilon+\gamma}}\left(1-s\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma}}P_n^{\left(2\sqrt{\varepsilon+\gamma},2\sqrt{\frac{1}{4}+\gamma}\right)}\left(1-2s\right)$$
(44)

Using the normalization condition, we obtain the normalization constant as follows

$$\int_0^\infty\left|\psi_{nl}(r)\right|^2dr=1 \quad (45)$$

By differentiating Eq.(19) with respect to r we have

$$\frac{ds}{dr}=-\alpha e^{-\alpha r} \quad (46)$$

Simplifying Eq.(46) and substituting Eq.(19) ,we obtain

$$dr=-\frac{ds}{\alpha s}$$
(47)

Substituting Eq.(47) into Eq.(45) and changing the limit we have,

$$\frac{1}{\alpha}\int_0^1\left|\psi_{nl}(s)\right|^2\frac{ds}{s}=1 \quad (48)$$

Let

$$y=1-2s \quad (49)$$

We differentiate Eq.(49) and simplify to obtain

$$ds=-\frac{dy}{2} \quad (50)$$

By simplifying Eq. (49) we have,

$$\left.s=\frac{1-y}{2},\ \frac{1}{s}=\frac{2}{1-y}\right\}$$
(51)

Substituting Eqs.(50) and (51) into Eq.(48) and changing the limit we obtain,

$$\frac{1}{\alpha}\int_{-1}^1\left|\psi_{nl}(y)\right|^2\frac{dy}{1-y}=1 \quad (52)$$

By substituting Eqs.(44) and (49) into Eq.(52) and with simple algebra we obtain

$$\frac{B_{nl}^2}{\alpha}\int_{-1}^{1}\left(\frac{1-y}{2}\right)^{2\sqrt{\varepsilon+\gamma}}\left(\frac{1+y}{2}\right)^{1+2\sqrt{\frac{1}{4}+\gamma}}\left[P_n^{\left(2\sqrt{\varepsilon+\gamma},2\sqrt{\frac{1}{4}+\gamma}\right)}y\right]^2 dy=1 \quad (53)$$

Let

$$\left.\mu=1+2\sqrt{\frac{1}{4}+\gamma},\ \mu-1=2\sqrt{\frac{1}{4}+\gamma},\ u=2\sqrt{\varepsilon+\gamma}\right\} \quad (54)$$

Substituting Eq.(54) into Eq.(53) we have

$$\frac{B_{nl}^2}{m_D(T)}\int_{-1}^{1}\left(\frac{1-y}{2}\right)^{u}\left(\frac{1+y}{2}\right)^{\mu}\left[P_n^{(2\mathrm{u},\mu-1)}y\right]^2 dy=1 \quad (55)$$

According to Onate et al.,[80] , integral of the form in Eq,(55) can be expresses as

$$\int_{-1}^{1}\left(\frac{1-p}{2}\right)^{x}\left(\frac{1+p}{2}\right)^{y}\left[P_n^{(2\mathrm{x},2y-1)}p\right]^2 dp=\frac{2\Gamma(x+n+1)\Gamma(y+n+1)}{n!x\Gamma(x+y+n+1)} \quad (56)$$

Hence, comparing Eq.(55) with the standard integral of Eq.(56), we obtain the normalization constant as

$$B_{nl}=\sqrt{\frac{n!u\alpha\Gamma(\mathrm{u}+\mu+\mathrm{n}+1)}{2\Gamma(u+n+1)\Gamma(\mu+n+1)}} \quad (57)$$

3. Results and Discussion

3.1 Discussion

We apply the data obtained from (Horchani*et al*., 2021) as presented in table 1, also adopted the conversion $\hbar c=1973.29$ eV $\mathring{A}$ (Horchani*et al*., 2021), with Eq.(38) when N = 3 to compute the vibrational energies of YP for CO, CH, N_2 and NO diatomic molecules as shown in Table 2. It is observed that for each vibrational quantum number, the vibrational energies increases with an increase in the rotational quantum number, for each of the chosen diatomic molecules.we numerically reported the energy eigenvalues for theYP by varying the principal quantum number n at a fixed orbital angular momentum quantum number l.

We also, computed the bound state energy eigenvalues of the YP using Eq.(38). We note that the energy increases as quantum number increases.The results showed good agreement with the earlier results of Ref.[23] with AIM, and Ref.[26] with SUSYQM.We have plotted the energy eigenvalues with the screening parameter, potential strength and quantum number as shown in Figs.2-4, for various values of quantum number. They plots show an increase in energy eigenvalues as the quantum number increases.In Fig. 2 we plotted the energy eigenvalues of aYP in the ground $n=0$ for different $l,(l=0,1,2,3,4)$ as a function of the screening parameter.From the plot, the energy eigenvalue inecreases.In Fig. 3, we plotted energy eigenvalues ofa YP as a function of the potential strengthin the ground $n=0$ for various l. The plot shows that the energy eigenvalues increase.In Fig. 4, we graphically show the variation of the YP in the ground $n=0$ for various l as a function of the principal quantum number. We observed that the particle ismore bounded in the ground state as the potential strength.

Table 1. Model parameters for some selected diatomic molecules in this study (Horchani*et al.*, 2021)

Molecules	$\alpha\left(\mathring{A}^{-1}\right)$	$\mu(amu)$
N_2	2.69860	7.003350
CO	2.29940	6.860586
NO	2.75340	7.468441
CH	1.52179	0.929931

Table 2. Bound state energy spectra $E_{nl}(eV)$ of YP for CO, CH, N_2 and NO diatomic molecules

n	l	E_{nl} (eV) of CO	E_{nl} (eV) of CH	E_{nl} (eV) of N_2	E_{nl} (eV) of NO
0	0	−59.25288028	−18.85027835	−73.23970978	−34.53518839
0	1	−59.25250501	−18.85025791	−73.23959572	−34.53518812
0	2	−59.25175467	−18.85021699	−73.23936765	−34.53518756
0	3	−59.25062968	−18.85015564	−73.23902562	−34.53518674
0	4	−59.24913067	−18.85007384	−73.23856971	−34.53518565
0	5	−59.24725845	−18.84997158	−73.23800004	−34.53518429
1	0	−59.32596985	−18.87811728	−73.37100834	−34.53969898
1	1	−59.32592581	−18.87809681	−73.37089269	−34.53969869
1	2	−59.32583776	−18.87805588	−73.37066144	−34.53969812
1	3	−59.32570568	−18.87799448	−73.37031465	−34.53969726
1	4	−59.32552965	−18.87791264	−73.36985235	−34.53969611
1	5	−59.32530968	−18.87781033	−73.36927468	−34.53969468
2	0	−59.40126038	−18.90604095	−73.50484846	−34.54441426
2	1	−59.40121530	−18.90602048	−73.50473127	−34.54441398
2	2	−59.40112517	−18.90597952	−73.50449695	−34.54441337
2	3	−59.40098997	−18.90591811	−73.50414554	−34.54441248
2	4	−59.40080973	−18.90583620	−73.50367709	−34.54441128
2	5	−59.40058454	−18.90573384	−73.50309178	−34.54440980
3	0	−59.47872659	−18.93404920	−73.64120398	−34.54933324
3	1	−59.47868051	−18.93402872	−73.64108535	−34.54933292
3	2	−59.47858833	−18.93398774	−73.64084805	−34.54933232
3	3	−59.47845012	−18.93392628	−73.64049220	−34.54933137
3	4	−59.47826584	−18.93384435	−73.64001786	−34.54933014
3	5	−59.47803559	−18.93374193	−73.63942518	−34.54932858
4	0	−59.55834376	−18.96214182	−73.78004941	−34.54932858
4	1	−59.55829667	−18.96212133	−73.77992938	−34.55445454
4	2	−59.55820256	−18.96208034	−73.77968920	−34.55445388
4	3	−59.55806138	−18.96201886	−73.77932912	−34.55445292
4	4	−59.55787323	−18.96193686	−73.77884909	−34.55445162

4	5	−59.55763814	−18.96183440	−73.77824929	−34.55445003
5	0	−59.55763814	−18.99031865	−73.92135979	−34.55977811
5	1	−59.64003961	−18.99029814	−73.92123836	−34.55977778
5	2	−59.63994361	−18.99025714	−73.92099551	−34.55977710
5	3	−59.63994361	−18.99019562	−73.92063132	−34.55977611
5	4	−59.63979964	−18.99019562	−73.92014586	−34.55977476
5	5	−59.63960774	−18.99001108	−73.91953921	−34.55977310

Table3: The bound state energy eigenvalues (in fm^{-1}) of Yukawa potential in units $\hbar = \mu = 1, \mathrm{N} = 3.$ Where $A_0 = \sqrt{2}$ and $\alpha = dA_0$ for comparison with other methods

State	D	Present work	SUSYQM[26]	AIM[23]
1s	0.002	-0.99600	-0.99601	-0.99600
	0.005	-0.99002	-0.99004	-0.99003
	0.010	-0.98014	-0.98015	-0.98014
	0.025	-0.95062	-0.95092	-0.95062
	0.050	-0.90363	-0.90363	-0.90363
2s	0.002	-0.24601	-0.24602	-0.24602
	0.005	-0.24014	-0.24015	-0.24014
	0.010	-0.23049	-0.23059	-0.23058
	0.025	-0.20355	-0.20355	-0.20355
	0.050	-0.16345	-0.16351	-0.16354
2p	0.002	-0.24601	-0.24602	-0.24601
	0.005	-0.24010	-0.24012	-0.24012
	0.010	-0.23048	-0.23049	-0.23049
	0.025	-0.20298	-0.20299	-0.20298
	0.050	-0.16115	-0.16114	-0.16148
3p	0.002	-0.10714	-0.10716	-0.10716
	0.005	-0.10143	-0.10142	-0.10141
	0.010	-0.09231	-0.09231	-0.09230
	0.025	-0.06814	-0.06814	-0.06815
	0.050	-0.03721	-0.03739	-0.03711
3d	0.002	-0.10714	-0.10715	-0.10715
	0.005	-0.10133	-0.10134	-0.10136
	0.010	-0.09201	-0.09202	-0.09212
	0.025	-0.06673	-0.06713	-0.06714
	0.050	-0.03361	-0.03388	-0.03383

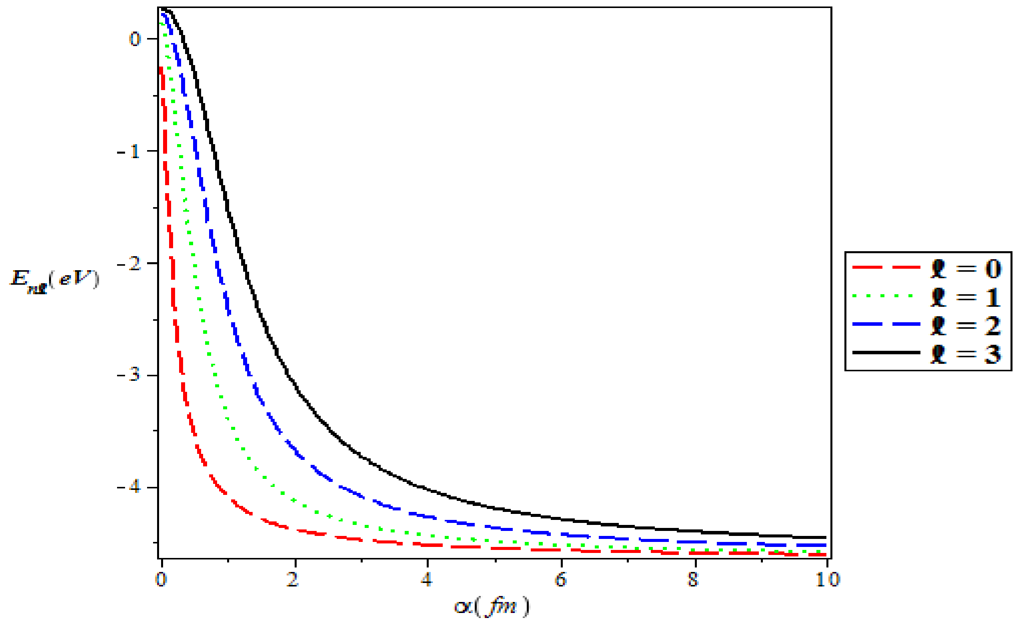

Figure2: Energy eigenvalues variation with screening parameter for various vibrational quantum numbers

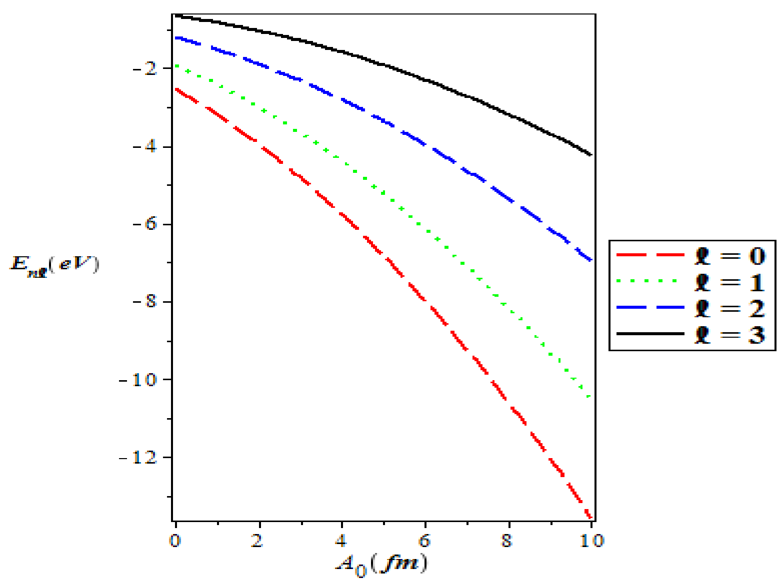

Figure3: Energy eigenvalues variation with potential parameter A_0 for various vibrational quantum numbers

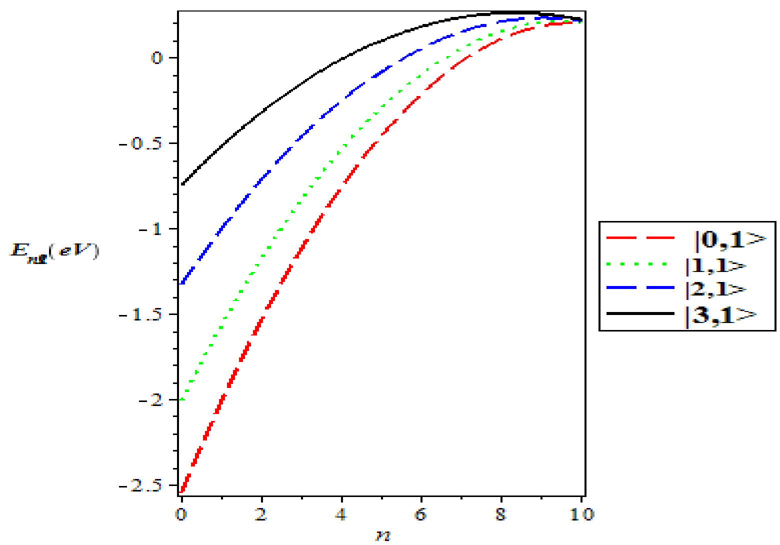

Figure 4: Energy eigenvalues variation with quantum number for various vibrational quantum number

Conclusions

In this work, we have obtainedthe bound state solutions of the Klein-Gordon equation for the Yukawa potential using the Nikiforov-Uvarov method. The energy eigenvalues are obtained both in the relativistic and non- relativistic regime and corresponding normalized wave function. We obtain a special case of Coulomb potential which agrees with Ref. [32] and Ref. [34] when $N = 3$. Also, the energy equation was applied to study some selected diatomic molecules. The bound state energy was obtained and comparison made with other works has proven the success of the formalism. The variation in the energy eigenvalues with screening parameter, potential strength and quantum number for various values of quantum number was also plotted. It was observed that the energy eigenvalues increases as the parameter increases.

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