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# First Order Multistep Method for Solving Diffusion Convection Equations Derivable from Polynomials as Basis Functions and its Applications

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## ABSTRACT

**In this work, a new numerical finite difference scheme for the solution of heat diffusion conduction equation arising from heat conduction is developed due to the recent growing interest in literatures in the derivation of continuous numerical finite difference method for solving heat diffusion convection equations. This was done based on the collocation and interpolation of the heat diffusion convection equations directly over multi steps along lines but without reduction to a system of Ordinary Differential Equations (ODE). The intention was to avoid the cost of solving a large system of coupled ODEs often arising from the reduction method by a semi - discretization. The performance of the new numerical finite difference scheme was tested. The numerical results obtained showed that the method provided the same results with the known explicit finite difference method. There was no semi-discretization involved in the derivation of this scheme, and no reduction of the heat diffusion convection equations to a system of ODE is recorded, but rather a system of algebraic equations is formulated. Therefore, the desire is to derive a new numerical scheme that will be used in finding the solutions of the system of algebraic equations formulated from the discretization of the heat diffusion convection equations with respect to the space and time variables. This new numerical method was applied to solve two different test problems with known explicit solutions by Schmidt.**

## 1. Introduction

There were some salient problems associated with the derivation of the known explicit method for solving PDE arising from heat diffusion convection equations. The earlier method used in the derivation of Schmidt method sought for the functional evaluation of some complex functions of heat diffusion convection equations, and also employs the Taylor series expansion. These functional evaluations and expansions might have introduced errors due to truncation (Biazar and Ebrahimi, 2005; Chawla and Katti, 1979; Yakubu et al., 2004). Based on that, there is of recent a growing interest in literatures to seek an alternative method of its derivation. For an efficient algorithm, therefore, there is the need to try to eliminate these problems. And algorithms that can overcome these problems will be of advantage (Crandall, 1995; Crane and Klopfenstein, 1965).

Following Awoyemi (2002), Crank (1947) and Onumanyi et al. (2002), a single PDE in one space variable, where $t$ and $x$ are the

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time and space coordinates respectively, and the quantities $h$ and $k$ are the mesh sizes in the space and time axes is considered. The interest is to extend the continuous numerical work of Sirisena et al. (2001); Dehghan (2003) to obtain another new continuous numerical method that can solve the equations arising from heat diffusion convection. This is done based on the collocation and interpolation of the equations directly over multi steps along lines but without reduction to a system of ODEs (Bao et al., 2003). The derivation avoids the cost of solving a large system of coupled ODEs often arising from the reduction method by a semi - discretization. The new method also helps eliminates the usual draw-back of stiffness arising in the conventional reduction method by semi-discretization as suggested by Awoyemi (1998) and Dieci (1992). The new scheme is applied to solve two different test problems with known explicit solutions by Schmidt. The results obtained are compared with the results from Schmidt method (see tables I & II), which clearly shows that the new method produced the same results with Schmidt. This method is of interest because its derivation avoided all the complexities that are inherent in the derivation of Schmidt method, including stiffness. The numerical results confirmed the efficacy and the validity of the new numerical scheme and suggested that it is an interesting and viable numerical tool.

## 1. Theoretical Method

### 1.1 The Solution Method

For such a new continuous numerical method to be developed, eqn. (1) with its associated initial and boundary conditions are used, according to Odekunle (2003):

We consider,

$$\frac{\partial U(x,t)}{\partial t} = V\frac{\partial^2 U(x,t)}{\partial x^2} + \overline{U}\,\frac{\partial U(x,t)}{\partial x},$$

(1)

In 2001, Yildiz & Subasi postulated that eqn. (1) is a special type of diffusion convection equation where $V, \overline{U} > 0$ are constants, defined for the interval $a < x < b$ and positive time with appropriate initial and boundary conditions. In setting up the solutions method therefore, we consider Adam & David (2002), and subdivide the interval $0 \le x \le b$ into $N$ equal subintervals by the grid points $x_m = mh, \quad m = 0,\ldots,N$ where $Nh = b$. We want to obtain an approximation $U(x,t)$ in the manner of Awoyemi (2003) and Saumaya et al., (2012) of the form

$$U(x,t) \approx U_K(x,t) = \sum_{r=0}^{k} a_r L_r(x,t)$$

(2)

$x_{i-1} \le x \le x_{i+1}$ and $r = 0(1)k+1$

Where $L_r(x,t)$ is the legendry's polynomials which are used as basis functions in the approximation, and $a_r$ are the parameters to be determined (Richard et al., 2001).

In 2003, Bao et al suggested that from the collocation equation (3),

$$U_K(x_m,t) = \sum_{r=0}^{k} a_r L_r(x_m,t$$

(3)

where $m = i-1, j, \quad i, j \ \ and \ \ i+1, j$

we can generate the following equations as follows:

$$\left.\begin{aligned} U_k(x_{i-1},t_j) &= a_o + a_1 x_{i-1} t_j + \ldots + a_k x^k{}_{i-1} t^k{}_j \approx U_{i-1,j} \\ U_k(x_i,t_j) &= a_o + a_1 x_i t_j + \ldots + a_k x^k{}_i t^k{}_j \approx U_{i,j} \\ U_k(x_{i+1},t_j) &= a_o + a_1 x_{i+1} t_j + \ldots + a_k x^k{}_{i+1} t^k{}_j \approx U_{i+1,j} \end{aligned}\right\}$$

(4)

Writing eqn. (4) as matrix in its augmented form, we have

$$\begin{bmatrix} 1 & x_{i-1}t_j & \dots & x^k{}_{i-1}t^k{}_j \\ 1 & x_i t_j & \dots & x^k{}_i t^k{}_j \\ 1 & x_{i+1}t_j & \dots & x^k{}_{i+1}t^k{}_j \end{bmatrix} \begin{bmatrix} a_0 \\ \dots \\ \dots \\ \dots \\ a_k \end{bmatrix} = \begin{bmatrix} U_{i-1,j} \\ U_{i,j} \\ U_{i+1,j} \end{bmatrix} \quad (5)$$

We solve eqn. (5) for the value of $a_k$ by Gaussian elimination method to obtain

$$a_k = \frac{U_{i+1,j} + U_{i-1,j} - U_{i,j}}{2ht^k{}_j}.$$

From eqn. (2) we equally generate eqn. (6) as follow:

$$U(x,t) = a_0 + a_1 xt + \dots + a_k x^k t^k \quad (6)$$

Putting the value of $a_k$ in eqn. (6), we obtain

$$U(x,t) = a_0 + a_1 xt + \dots + x^k t^k \left( \frac{U_{i+1,j} + U_{i-1,j} - 2U_{i,j}}{2ht^k{}_j} \right) \quad (7)$$

By Awoyemi (2002) and Zheyin & Qiang (2012), we take the first and second derivatives of equation (7) with respect to $x$

$$U'(x,t) = a_1 t + \dots + kx^{k-1}t^k \left( \frac{U_{i+1,j} + U_{i-1,j} - 2U_{i,j}}{2ht^k{}_j} \right)$$

and obtain

$$U_i''(x,t) = k(k-1)x^{k-2}t^k \left( \frac{U_{i+1,j} + U_{i-1,j} - 2U_{i,j}}{2ht^k{}_j} \right) \quad (8)$$

Again, by Benner and Mena (2004), we consider the collocation equations

$$U_K(x_n,t) = \sum_{r=0}^{k} a_r L_r(x_n,t) \quad (9)$$

with $n = i-1, j \ and \ i+1, j$

We can generate eqn. (10) from eqn. (9) as follows:

$$\left. \begin{aligned} U_k(x_{i-1},t_j) &= a_o + a_1 x_{i-1}t_j + \dots + a_k x^k{}_{i-1}t^k{}_j \approx U_{i-1,j} \\ U_k(x_{i+1},t_j) &= a_o + a_1 x_{i+1}t_j + \dots + a_k x^k{}_{i+1}t^k{}_j \approx U_{i+1,j} \end{aligned} \right\} \quad (10)$$

Writing eqn. (10) as matrix in its augmented form, we have

$$\begin{bmatrix} 1 & x_{i-1}t_j & \dots & x^k{}_{i-1}t^k{}_j \\ 1 & x_{i+1}t_j & \dots & x^k{}_{i+1}t^k{}_j \end{bmatrix} \begin{bmatrix} a_0 \\ \dots \\ \dots \\ \dots \\ a_k \end{bmatrix} = \begin{bmatrix} U_{i-1,j} \\ U_{i+1,j} \end{bmatrix} \quad (11)$$

Solving eqn. (11) for $a_k$ we obtain

$$a_k = \frac{U_{i+1,j} - U_{i-1,j}}{2ht^k{}_j}$$

, but from eqn. (1) we obtain

$$U(x,t) = a_0 + a_1 xt + \dots + a_k x^k t^k \quad (12)$$

Putting the value of $a_k$ in eqn. (12) we obtain

$$U(x,t) = a_0 + a_1 xt + \dots + x^k t^k \left( \frac{U_{i+1,j} - U_{i-1,j}}{2ht^k{}_j} \right) \quad (13)$$

Again, by Awoyemi (2002), we take the first derivative of eqn. (13) with respect to $x$ and obtain

$$U'(x,t) = k\left(x^{k-1}t^k\right)\left( \frac{U_{i+1,j} - U_{m-1,j}}{2ht^k{}_j} \right) \quad (14)$$

Similarly, by Bensoussan et al (2007) we consider and interchanging the roles of $x \ and \ t$ to obtain an approximation of the form

$$U(t,x) \approx U_k(t,x) = \sum_{r=0}^{k} a_r L_r(t,x) \quad (15)$$

where $r = 0(1)k + 1$

Again, from the collocation equation, we obtain

$$U_K(x,t_g) = \sum_{r=0}^{k} a_r L_r(x,t_g) \quad (16)$$

We make $g = \quad i, j \;\; and \;\; i, j+1$

By substituting the values of $g$ in eqn. (16) we obtain eqn. (17) as follows:

$$\left.\begin{aligned} U_k(t_j, x_i) &= a_o + a_1 t_j x_i + \ldots + a_k t^k{}_j x^k{}_i \approx U_{i,j} \\ U_k(t_{j+1}, x_i) &= a_o + a_1 t_{j+1} x_i + \ldots + a_k t^k{}_{j+1} x^k{}_i \approx U_{i,j+1,} \end{aligned}\right\} \quad (17)$$

Writing eqn. (17) as matrix in its augmented form, we have

$$\begin{bmatrix} 1 & x_i t_j & \ldots x^k{}_i t^k{}_j \\ 1 & x_i t_{j+1} & \ldots x^k{}_i t^k{}_{j+1} \end{bmatrix} \begin{bmatrix} a_0 \\ \ldots \\ \ldots \\ \ldots \\ a_k \end{bmatrix} = \begin{bmatrix} U_{i,j} \\ U_{i,j+1} \end{bmatrix} \quad (18)$$

We solve the matrix eqn. (18) for $a_k$ by using Gaussian elimination method, we obtain

$$a_k = \frac{U_{i,j+1} - U_{i,j}}{2hx^k{}_j}$$

Putting $r = 0, 1$ in eqn. (2) and manipulating mathematically, we obtain

$$U(t,x) = a_0 + a_1 xt + \ldots + a_k x^k t^k \quad (19)$$

Substituting for the value of $a_k$ in eqn. (19) we have

$$U(t,x) = a_0 + a_1 xt + \ldots + x^k t^k \left( \frac{U_{i,j+1} - U_{i,j}}{2hx^k{}_i} \right) \quad (20)$$

By following Biazar and Ebrahimi (2005) and Pierre (2008), we take the first derivative of eqn. (20) with respect to $t$. And we have

$$U_j{}'(t,x) = .kt^{k-1} x^k \left( \frac{U_{i,j+1} - U_{i,j}}{2hx^k{}_i} \right), \quad (21)$$

Again, by considering Dehghan (2003) and Penzl (2000), we collocate eqn. (8) and eqn. (14) at $x = x_i$, and eqn. (21) at $t = t_j$ and substituting the resulting equations in eqn. (2), we obtain a new numerical scheme which solves explicitly the diffusion convection equations numerically.

To illustrate this method therefore, we use our new continuous numerical scheme to solve two test problems (1) and (2) respectively with known exact solutions. The results obtained is compared with the exact and the solutions obtained from Schmidt method as shown below in Tables 1& 2.

## 3. Application of the Solution Method

### 3.1. Specific Problem

Example 1

Use the scheme to approximate the solution to the diffusion convection equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + 2\frac{\partial U}{\partial x} \qquad 0 < x < 20,\ 0 < t < 20$$

$$U(x,0) = |10 - x|,\ 0 \le x \le 20$$

$$U(0,t) = U(20,t) = 0, \qquad 0 \le t \le 20$$

**Table 1: Results of action of the new scheme on problem 1.**

| $x$ | Analytical solutions $U(x,t)$ | Solution from New Method $U(x,t)$ | Solution from Finite Difference Method(Schmidt) $U(x,t)$ | Error from Schmidt Method | Error from New method |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0.380814721 | 0.380862804 | 0.380862804 | 4.8 x E-5 | 4.8 x E-5 |
| 0.50 | 0.703723471 | 0.703742699 | 0.703742699 | 1.9 x E-5 | 1.9 x E-5 |
| 0.75 | 0.919471568 | 0.919484148 | 0.919484148 | 1.3 x E-5 | 1.3 x E-5 |
| 1.00 | 0.995167871 | 0.99524247 | 0.99524247 | 7.5 x E-5 | 7.5 x E-5 |
| 1.25 | 0.919471568 | 0.919484148 | 0.919484148 | 1.3 x E-5 | 1.3 x E-5 |
| 1.5 | 0.703723471 | 0.703742699 | 0.703742699 | 1.9 x E-5 | 1.9 x E-5 |
| 1.75 | 0.380814721 | 0.380862804 | 0.380862804 | 4.8 x E-5 | 4.8 x E-5 |
| 2.00 | 0 | 0 | 0 | 0 | 0 |

Example 2

Use the scheme to approximate the solution to the diffusion convection equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + 2\frac{\partial U}{\partial x} \qquad 0 < x < 20,\ 0 < t < 20$$

$$U(x,0) = \frac{x}{2},\ 0 \le x \le 20$$

$$U(0,t) = 0,\ U(20,t) = 0, \qquad 0 \le t \le 20$$

**Table 2: Results of action of the new scheme on problem 2.**

| $x$ | Analytical Solution $U(x,t)$ | Solution from Finite Difference method (Schmidt) $U(x,t)$ | Solution from New Method $U(x,t)$ | Error from Finite Difference Method (Schmidt) | Error from New Method |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0.10 | 0.30806537 | 0.30807172 | 0.30807172 | 6.3 X E-6 | 6.3 X E-6 |
| 0.20 | 0.585975167 | 0.58598723 | 0.58598723 | 1.20 X E-5 | 1.20 X E-5 |
| 0.30 | 0.806525626 | 0.80654224 | 0.80654224 | 1.7 X E-5 | 1.7 X E-5 |
| 0.40 | 0.948127737 | 0.948147264 | 0.948147264 | 2.0 X E-5 | 2.0 X E-5 |
| 0.50 | 0.99692050 | 0.996941032 | 0.996941032 | 2.1 X E-5 | 2.1 X E-5 |
| 0.60 | 0.948127737 | 0.948147264 | 0.948147264 | 2.0 X E-5 | 2.0 X E-5 |
| 0.70 | 0.806525626 | 0.80654224 | 0.80654224 | 1.7 X E-5 | 1.7 X E-5 |
| 0.80 | 0.585975167 | 0.58598723 | 0.58598723 | 1.20 X E-5 | 1.20 X E-5 |
| 0.90 | 0.30806537 | 0.30807172 | 0.30807172 | 6.3 X E-6 | 6.3 X E-6 |
| 1.00 | 0 | 0 | 0 | 0 | 0 |

### 3.2 Discussion of Results

The action of our new scheme on test problems 1 and 2 have shown that it produces the same results with the known Schmidt method and near exact solutions (see tables 1 & 2). The advantage of this new continuous numerical scheme over the old Schmidt method, therefore, is its easy of derivation and presumably in terms of stability which I intend to investigate in my next research. If the stability of this new approach turns to be better than that of the Schmidt, then it will confirm that this is a good and valid scheme in the hands of researchers for deployment.

I invite the teeming research community to join me in trying to investigate the stability of this new continuous numerical method when applied to solving heat diffusion convection equations arising from heat diffusion. Also, we challenge our researchers to come – up with other new scheme with higher fractional mesh points, and check whether they perform better than this our new method or not. Also, we recommend that higher orders of this method be investigated by other researchers. We expect some exact values will be obtained at the end of the day, because it appears as if the higher the order the more accurate the results become. This should be investigated thoroughly. Also, recommended for consideration are the existence and convergence of this new approach.

## Conclusion

A numerical method which involves the discretization of the diffusion convection equations to a system of algebraic equations is formulated for solving the diffusion convection problems. To check the accuracy of the numerical method, we follow the work of Eyaya (2010) and apply the new scheme to solve two different test problems with known Schmidt and exact solutions. The overall numerical results obtained showed that the performance of our scheme is exactly the same with that of Schmidt finite difference method and almost analytical solutions. These confirmed the validity of the new numerical scheme and suggested that it is an interesting and viable numerical method.

## References

[1] Adam, A. and David, R., (2002). One dimensional heat equation. http://www.ng/online.redwoods.cc.ca.us/instruct/darnold/deproj/sp02/.../paper.pdf

[2] Awoyemi, D. O., (2002). An Algorithmic collocation approach for direct solution of special fourth – order initial value problems of ordinary differential equations. *Journal of the Nigerian Association of Mathematical Physics*, **6**: 271 – 284.

[3] Awoyemi, D. O., (2003). A p – stable linear multistep method for solving general third order Ordinary differential equations. *Int. J. Computer Math*., **80** (8):987 - 993.

[4] Bao, W., Jaksch, P. and Markowich, P.A., (2003). Numerical solution of the Gross – Pitaevskii equation for Bose – Einstein condensation. *J. Compt. Phys*., **187**(1): 318- 342.

[5] Benner, P. and Mena, H., (2004). BDF methods for large scale differential Riccati equations in proc. of mathematical theory of network and systems. *MTNS.* Edited by Moore, B. D., Motmans, B., Willems, J., Dooren, P.V. & Blondel, V.

[6] Bensoussan, A. Da Prato, G., Delfour, M. and Mitter, S., (2007). Representation and control of infinite dimensional systems. 2nd edition.

[7] Birkhauser: Boston, MA. Motmans, B., Willems, J., Dooren, P. V., Blondel, V., Biazar, J. and Ebrahimi, H., (2005). An approximation to the solution of hyperbolic equation by a domain decomposition method and comparison with characteristics method. *Appl. Maths. and Comput*., **163**:633 - 648.

[8] Dehghan, M., (2003). Numerical solution of a parabolic equation with non – local boundary specification. *Appl. Maths. Comput*., **145**:185 – 194.

[9] Eyaya, B. E., (2010). Computation of the matrix exponential with application to linear parabolic PDEs. http://www.dip.sun.ac.za/~eyaya/PGD-Essay-Template-2009_10.pdf

[10] Penzl, T., (2000). Matrix analysis. *SIAM J.*, ***21***: 1401- 1418.

[11] Pierre, J. (2008). Numerical solution of the dirichlet problem for elliptic parabolic equations. SIAM J. *Soc. Indust. Appl. Maths.*, **6**(3): 458 – 466.

[12]Richard, L., Burden, J. and Douglas, F., (2001). *Numerical analysis*. Seventh ed., Berlin: Thomson Learning Academic Resource Center.

[13]Saumaya, B., Neela, N. and Amiya, Y. Y., (2012). Semi discrete Galerkin method for equations of Motion arising in Kelvin – Voitght model of viscoelastic fluid flow. *Journal of Pure and Applied Sciences,* 3 (2 & 3): 321- 343.

[14]Yildiz, B. and Subasi, M., (2001). On the optimal control problem for linear Schrodinger equation. *Appl. Maths. and Comput.,*121:373-381.

[15]Zheyin, H. R. and Qiang, X. (2012). An approximation of incompressible miscible displacement in porous media by mixed finite elements and symmetric finite volume element method of characteristics. *Applied Mathematics and Computation,* Elsevier, 143: 654 - 672.