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Variational Iteration Decomposition method for Numerical Solution of Boundary Value Problems with Mamadu-Njoseph Polynomials

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Abstract
In this paper, we seek the numerical solution of boundary value problems through an elegant mixture of the variational iteration method (VIM) and Adomian decomposition method (ADM) called the Variational iteration decomposition method (VIDM). In VIDM, the correction functional is first constructed and the general Lagrange multiplier is calculated. The initial approximation is uniquely determined by employing Mamadu-Njoseh polynomials as ansatz (test) functions satisfying the prescribed condition at the lower boundary at $x = a$ in the interval $[a, b]$. In the nonlinear problems, the nonlinear terms are substituted with Adomian polynomials. The method is applied to selected linear and nonlinear boundary value problems of fifth order and the results obtained show that the method converges rapidly to the exact solution with few iterations. Results obtained are presented in graphs and in tables. Maple 18 software was used in executing all computational frameworks.

Keywords: Mamadu-Njoseh polynomials, variational iteration, boundary value problem.

1.0 Introduction

We consider the general boundary value problem of the form (Njoseh and Mamadu, 2016a)

$$\sum_{r=0}^{n} f_r(x) y^{(n-r)}(x) = g(x), \quad a < x < b, \tag{1}$$

with the initial conditions

$$y^{(k)}(a) = A_k, \quad y^{(k)}(b) = B_k, \quad k = 0,1,2,\ldots,n, \tag{2}$$

where $f_r(x)$, $r = 0(1)n$, $y(x)$, and $g(x)$ are real and continuous on $[a, b]$, A_k, B_k, $k = 0(1)n$, are real constants and n is the order of equation (1).

In recent years, equation (1) has been of keen interest to researchers due to its significance in the mathematical modelling of real life situations. Consequently, several methods (iterative) have being proposed and implemented by various researchers. These iterative methods produced approximate solutions to the analytic solution. Examples of such methods include, Tau-collocation method (Mamadu and Njoseh, 2016a), power series approximation method (PSAM) (Njoseh and Mamadu, 2016b), the

Galerkin method (Mamadu and Njoseh, 2016b), transformed generate approximation method (TGAM) (Njoseh and Mamadu, 2016c), variational iteration method (VIM) (Abbasbandy and Shivanian, 2009; Mamadu and Njoseh, 2016c), Adomian decomposition method (ADM) (Adomian and Rach, 1992; Adomian, 1994; Biazer, 2005; Wazwaz, 2011), weighted residual method (WRM) (Oderinu, 2014), differential transform method (DTM) (Islam *et al*, 2009), B-spline function method (Caglar *et al*, 1999), etc. This paper adopts the method of variational iteration decomposition method (VIDM) to seek the numerical solution of the fifth order boundary value problem using Mamadu-Njoseh polynomials as ansatz functions. The method merges the VIM and ADM to give VIDM. In VIDM, the correction functional is first constructed and the general Lagrange multiplier is calculated. The initial approximation is uniquely determined by employing Mamadu-Njoseh polynomials as ansatz functions satisfying the prescribed condition at the lower boundary at $x = a$. Also, the nonlinear terms are substituted with Adomian polynomials (Adomian and Rach, 1992; Adomian, 1994; Biazer, 2005; Wazwaz, 2011). The method is highly explicit, accurate and straightforward and converges rapidly to the analytic solution with few iterations. Numerical results obtained are compared with results in (Njoseh and Mamadu, 2016c) as available in literature.

2.0 Theoretical Methods

2.1 Mamadu-Njoseh Polynomials

Given the weight function

$$w(x) = x^2 + 1, x \in [-1,1], \tag{3}$$

the formulation of Mamadu-Njoseh polynomials $\varphi_n, n = 0, 1, 2, 3, \dots,$ are based on these three properties;

i. $\varphi_n(x) = \sum_{i=0}^{n} C_i^{(n)} x^i$

ii. $< \varphi_m(x), \varphi_m(x) >= \int_{-1}^{1} w(x)\varphi_m(x)\varphi_n(x)dx = 0, \quad m \neq n$

iii. $\varphi_n(x) = 1$

Hence, the first seven Mamadu-Njoseh polynomials are outlined below as reported in (Njoseh and Mamadu, 2016a; Mamadu and Njoseh, 2016b)

$$\varphi_0(x) = 1$$
$$\varphi_1(x) = x$$
$$\varphi_2(x) = \frac{1}{3}(5x^2 - 2)$$
$$\varphi_3(x) = \frac{1}{5}(14x^3 - 9x)$$
$$\varphi_4(x) = \frac{1}{648}(333 - 2898x^2 + 3213x^4)$$
$$\varphi_5(x) = \frac{1}{136}(325x - 1410x^3 + 1221x^5)$$
$$\varphi_6(x) = \frac{1}{1064}(-460 + 8685x^2 - 24750x^4 + 17589x^6)$$

However, if the interval $[-1,1]$ is mapped bijectively to any other interval $[a, b]$, then

$$\varphi_n^*(x) = \sum_{i=0}^{n} C_i^{(n)} x^i = \left(\frac{2x-a-b}{b-a}\right), \quad x \in [a,b], \qquad (4)$$

where $\varphi_i^*(x),\ i = 0,1,2,3,\ldots,$ are called the shifted Mamadu-Njoseh polynomials.

2.2 Variational Iteration Method (VIM)

We consider equation (1) with the prescribed auxiliary boundary. The VIM (Abbasbandy and Shivanian, 2009; Mamadu and Njoseh, 2016c) requires the construction of correction functional for equation (1) as follows:

$$y_{k+1}(x) = y_k(x) + \int_0^x \lambda(s)\left(\frac{d^n}{ds^5}\left(\sum_{r=0}^{n} f_r(s) y_k^{(n-r)}(s)\right) - g(s)\right) ds, k \geq 0, \qquad (5)$$

where $\lambda(s)$ is the general Lagrange multiplier and can be obtained using generalized formula (Abbasbandy and Shivanian, 2009)

$$\lambda_n(s) = (-1)^n \frac{(s-x)^{(n-1)}}{(n-1)!}, \qquad (6)$$

where n in equation (6) is the highest occurring derivative in (1).

2.3 Adomian Decomposition Method (ADM)

The ADM (Adomian and Rach, 1992; Adomian, 1994; Wazwaz, 2011) requires that the solution to equation (1) be defined as

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \qquad (7)$$

and the nonlinear term, say $Ny(x)$ is decomposed as

$$Ny(x) = \sum_{n=0}^{\infty} A_n, \qquad (8)$$

where $A_n\ n \geq 0,$ are the Adomian polynomials which are determined using the algorithms (Adomian and Rach, 1992;

Adomian, 1994; Abbasbandy and Shivanian, 2009; Wazwaz, 2011)

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)\right]_{\lambda=0}. \quad (9)$$

If we let $Ny(x) = f(x)$, then first few Adomian polynomials are as follows:

$$A_0 = f_0(x),\ A_1 = f_1(x)f_0'(x),\ A_2 = f_2(x)f_0'(x) + \frac{f_1^{\prime}(x)}{2!}f_0''(x) + \cdots \quad (10)$$

2.4 Variational Iteration Decomposition Method (VIDM)

The VIDM entails an elegant mixture of VIM and ADM. There are two cases to be considered.

Case I: If equation (1) embodies nonlinear terms, the VIDM requires that the nonlinear terms be redefined with equation (8) in our iterative scheme in equation (5).

Case II: If equation (1) is surely linear, then the linear terms are uniquely redefined using equation (7) in our iterative scheme in equation (5).

2.4.1 Determination of Initial Approximation

To implement our iterative scheme in the above cases, we need an initial approximation to start our iteration. Many scholars at this stage formulate their initial approximation by satisfying the analytic solution with the given boundary conditions. However, our case in this paper is different and unique. We make an estimate of the initial approximation by employing Mamadu-Njoseh polynomials as ansatz functions. These polynomials have been discussed in section 2.1 of this work.

Let the initial approximation of (1) be given as

$$y(x) = \sum_{i=0}^{n-1} \alpha_i \varphi_i(x), \quad (11)$$

where $\alpha_i, i = 0(1)(n-1)$, are parameters unknown and n is the order of equation (1).

Now, transforming (1) and (2) into system of ordinary differential equations, we have

$$y = y_1,\ \frac{dy}{dx} = y_2,\ \frac{d^2y}{dx^2} = y_3,\ \ldots,\ \frac{d^{(m)}y}{dx^{(m)}} = \frac{-g(x)}{\sum_{r=0}^{n} f_r(x)}, \quad (12)$$

$m = n - r,\ r = 0, 1, 2, 3, \ldots,$ and n is the order of (1).

For instance, if we let $n = 2$ in (11), we have

$$y(x) = \alpha_0\varphi_0(x) + \alpha_1\varphi_1(x). \quad (13)$$

Substituting (13) into (12) at the boundary $x = a$, we have the system of linear equations

$$\begin{pmatrix} \varphi_0(a) & \varphi_1(a) \\ \varphi_0'(a) & \varphi_1'(a) \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}. \quad (14)$$

Solving equation (14) yields the results

$$\left.\begin{aligned} \alpha_0 &= \frac{A_0\varphi_1'(a)-A_1\varphi_1(a)}{\varphi_0(a)\varphi_1'(a)-\varphi_1(a)\varphi_0'(a)} \\ \alpha_1 &= \frac{A_1\varphi_0(a)-A_0\varphi_0'(a)}{\varphi_0(a)\varphi_1'(a)-\varphi_1(a)\varphi_0'(a)} \end{aligned}\right\} \quad (15)$$

Substituting (15) into (13) to obtain the initial condition as

$$y(x) = \sum_{i=0}^{1} \frac{A_i}{i!} x^i. \quad (16)$$

Continuing this process, we obtain the general matrix equation

$$Ax = b \quad (17)$$

where

$$A = \begin{pmatrix} \varphi_0(a) & \varphi_1(a) & \varphi_2(a) & \cdots & \varphi_{n-1}(a) \\ 0 & \varphi_1'(a) & \varphi_2'(a) & \cdots & \varphi_{n-1}'(a) \\ 0 & 0 & \varphi_2''(a) & \cdots & \varphi_{n-1}''(a) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$x = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{n-1})^T,$$

$$b = (A_0, A_1, A_2, \cdots, A_{n-1})^T.$$

Solving the matrix equation (17) for $\alpha_i, i = 0(1)(n-1)$, with the aid of Maple 18 software and substituting back in equation (11), we have our initial approximation to equation (1) as

$$y(x) = \sum_{i=0}^{n-1} \frac{A_i}{i!} x^i, \quad (18)$$

where $A_i, i = 0, 1, 2, \dots, (n-1)$, are real constants in $[a, b]$.

Thus, the VIDM scheme for a generalized boundary value problem with Mamadu-Njoseh polynomials as ansatz functions in either case above is given as

$$y_0(x) = \sum_{i=0}^{n-1} \frac{A_i}{i!} x^i,$$

$$y_{k+1}(x) = y_k(x) + \int_0^x \lambda(s) \left(\frac{d^n}{ds^5} \left(\sum_{r=0}^{n} f_r(x) y_k^{(n-r)}(x) \right) - g(x) \right) ds, k \geq 0. \quad (19)$$

The unknown parameter in each iteration is determined at the other boundary condition $x = b$.

3.0 Results and Discussion

3.1 Results (Numerical Examples)

We now give numerical illustrations using VIDM with Mamadu-Njoseh polynomials as the ansatz functions. Basically, we apply the iterative scheme generated in equation (15) to fifth order boundary value problems, that is, $n = 5$ in equation (1). Numerical results obtained are compared in terms of their maximum errors attained as available in literature (Njoseh and Mamadu, 2016c).

Example 3.1

Consider the fifth order nonlinear BVP of the form (Njoseh and Mamadu, 2016c)

$$y^{(v)}(x) = e^{-x}y^2(x) \quad (20)$$

subject to the boundary conditions

$$y(0) = y'(0) = 0, \; y''(0) = 1, \; y(1) = y'(1) = e. \quad (21)$$

The exact solution is

$$y(x) = e^x. \quad (22)$$

Using the VIDM scheme for (19) with Mamadu-Njoseh polynomials as in equation (15) for $n = 5$, at $x = 0$, we have

$$y_0(x) = \sum_{i=0}^{4} \frac{A_i}{i!}x^i = 1 + x + \frac{1}{2}x^2 + \frac{A_3}{6}x^3 + \frac{A_4}{24}x^4, \quad (23)$$

$$y_{k+1}(x) = y_k(x) + \int_0^x \lambda(s)\left(\frac{d^5}{ds^5}y_k(s) - e^{-x}y_k^2(s)\right)ds, \; k \geq 0 \quad (24)$$

We define

$$e^{-x} \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

and

$$\lambda(s) = \frac{(s-x)^4}{24}$$

(from equation (6)) such that equation (24) becomes

$$y_{k+1}(x) = y_k(x) - \int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}y_k(s) - \left(1 - s + \frac{1}{2}s^2 - \frac{1}{3!}s^3 + \frac{1}{4!}s^4\right)y_k^2(s)\right)ds, \quad k \geq 0 \quad (25)$$

Since, equation (20) is nonlinear, equation (25) can be written as

$$y_{k+1}(x) = y_k(x) - \int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}y_k(s) - \left(1 - s + \frac{1}{2}s^2 - \frac{1}{3!}s^3 + \frac{1}{4!}s^4\right)\sum_{n=0}^{\infty} A_n(s)\right)ds, \; k \geq 0. \quad (26)$$

where $A_n, n \geq 0$, are the Adomian polynomials defined in equation (9).

Now, for $k = 0$,

$$y_1(x) = y_0(x) - \int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}y_0(s) - \left(1 - s + \frac{1}{2}s^2 - \frac{1}{3!}s^3 + \frac{1}{4!}s^4\right)A_0(s)\right)ds. \quad (27)$$

where

$$A_0(s) = y_0^2(s).$$

For $k = 1$,

$$y_2(x) = y_1(x) - \int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}y_1(s) - \left(1 - s + \frac{1}{2}s^2 - \frac{1}{3!}s^3 + \frac{1}{4!}s^4\right)A_1(s)\right)ds. \quad (28)$$

where

$$A_0(s) = 2y_0(s)y_1(s).$$

$$y_k(x) = y_{k-1}(x) - \int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}y_{k-1}(s) - \left(1 - s + \frac{1}{2}s^2 - \frac{1}{3!}s^3 + \frac{1}{4!}s^4\right)A_{k-1}(s)\right)ds, \; k \geq 1 \quad (29)$$

Solving the above equations (23), (27)-(28) for the constant A_3 and A_4 at $x = 1$, and substituting back in (23), (27)-(28), yields the results presented **table 1** below.

Table 1: Values of A_3 and A_4 with the maximum errors for each iterate for Example 3.1

	A_3	A_4	**Maximum Error**
$y_0(x)$	1.000011	0.999953	4.2820E-06
$y_1(x)$	0.857561	2.047673	9.5000E-08
$y_2(x)$	0.639535	3.639544	7.2000E-08

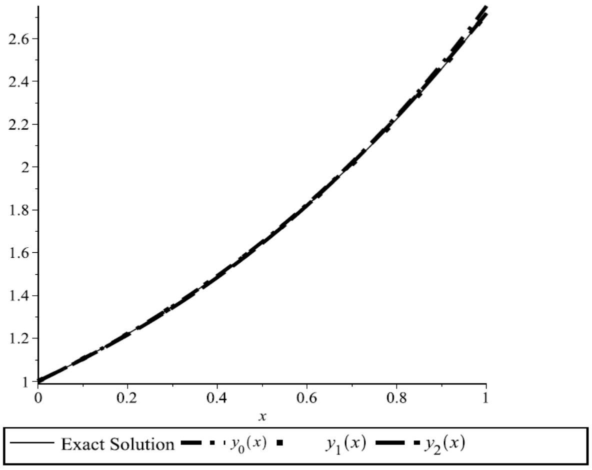

Figure 1: Comparison of iterates and the exact solution for Example 3.1

Example 3.2

Consider the fifth order linear BVP of the form (Njoseh and Mamadu, 2016a; Njoseh and Mamadu, 2016c)

$$y^{(v)}(x) = y - 15e^x - 10xe^x, \tag{30}$$

subject to the boundary conditions

$$y(0) = 0,\ y'(0) = 1, y''(0) = 0,\ y(1) = 0, y'(1) = -e. \tag{31}$$

The exact solution is

$$y(x) = x(1-x)e^x. \tag{32}$$

Using the VIDM scheme for (30) with Mamadu-Njoseh polynomials in equation (15) for $n = 5$, at $x = 0$, we have

$$y_0(x) = x + \frac{A_3}{6}x^3 + \frac{A_4}{24}x^4, \tag{33}$$

$$y_{k+1}(x) = y_k(x) - \int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}\left(\sum_{n=0}^{\infty} y_n(s)\right) - \sum_{n=0}^{\infty} y_n(s) + (15+10s)\left(1 - s + \frac{1}{2}s^2 - \frac{1}{3!}s^3 + \frac{1}{4!}s^4\right)\right)ds. \tag{34}$$

Now, for $k = 0$,

$$y_1(x) = y_0(x) - \int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}y_0(s) - y_0(s) + (15+10s)\left(1 - s + \frac{1}{2}s^2 - \frac{1}{3!}s^3 + \frac{1}{4!}s^4\right)\right)ds. \tag{35}$$

For $k = 1$,

$$y_2(x) = y_1(x) - \int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}y_1(s) - y_1(s) + (15+10s)\left(1 - s + \frac{1}{2}s^2 - \frac{1}{3!}s^3 + \frac{1}{4!}s^4\right)\right)ds. \tag{36}$$

$\vdots$

$$y_k(x) = y_{k-1}(x)$$

$$-\int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}y_{k-1}(s) - y_{k-1}(s) + (15+10s)\left(1 - s + \frac{1}{2}s^2 - \frac{1}{3!}s^3 + \frac{1}{4!}s^4\right)\right)ds, k \geq 1$$

. (37)

Solving the above equations (33), (35) - (36) for the constant A_3 and A_4 at $x = 1$, and substituting back in (33), (35)-(36), yields the results presented in **table 2** below.

Table 2: Values of A_3 and A_4 with the maximum errors for Example 3.2

	A_3	A_4	**Maximum Error**
$y_0(x)$	-1.690309	-17.238764	1.8107E-04
$y_1(x)$	-2.999585	-8.001926	6.1200E-08
$y_2(x)$	-2.999763	-8.001098	3.4960E-08

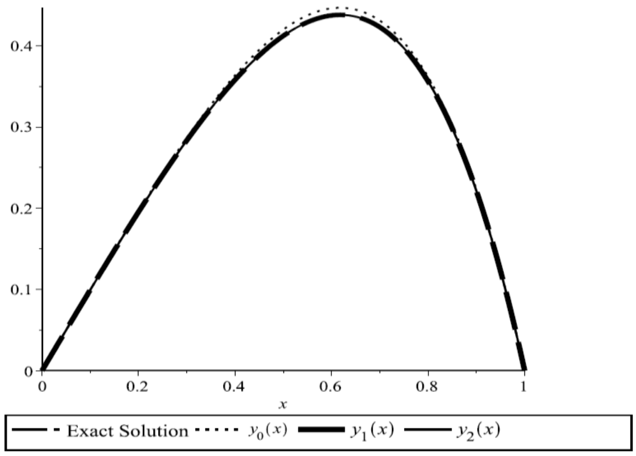

Figure 2: Comparison of iterates and the exact solution for Example 3.2

3.2 Discussion of Results

The Mamadu-Njoseh polynomials have been successively applied as ansatz functions in the numerical solution of fifth order boundary value problems in a variational iteration decomposition method. As implemented for example 6.1 and 6.2, the maximum errors obtained are of order 10^{-8} for both problems. On the other hand, the maximum error in (Njoseh and Mamadu, 2016c) for example 6.1 and 6.2 are of order 10^{-6} and 10^{-8} respectively with Chebychev polynomials. It is also evident that the approximate solution converges rapidly to the exact solution with few iterations as shown in Figures 1 and 2.

4.0 Conclusion

The VIDM involving Mamadu-Njoseh polynomials as ansatz solutions has been applied to linear and nonlinear boundary value problems. The numerical evidences obtained show that the method is accurate and having an excellent rate of convergent. The orthogonality property of the Mamadu-Njoseh polynomials is an added advantage for the rapid convergent of the iterative scheme. This implies that the Mamadu-Njoseh polynomials can serve as ansatz functions in the numerical solution of any higher order boundary value problem and other differential equations.

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