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Improved Elzaki Transform Decomposition Method for the Analytic Solution of Gas Dynamic Equations

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Abstract
In this paper, we implement the improved Elzaki transform decomposition method for the analytic solution of gas dynamic problems. The method merges the properties of the Elzaki Transform method and Adomian decomposition method. The method requires no additional work stress such as perturbation, linearization or discretization and proves very effective in resolving the gas dynamic equations. The resulting numerical evidences show that the method converges favourably to the analytic solution. All computations are implemented with maple 18 software.

Keywords: Elzaki transform method, Adomian decomposition method, Partial differential equation, Gas dynamic equation

Introduction
The business of modeling in mathematics cuts across every field of science and technology. The gas dynamic equation is one of such models that find itself in abrasive blasting, gas pipeline, aircraft, jet engine, rocket motors, etc. As such, resolving this equation is of keen interest to scientists. Available analytic methods for these equations are often restricted and complex in evaluation. In this regard, numerical methods have been deemed more efficient and reliably available in resolving the gas dynamic equations. Popular available numerical methods include the Adomian decomposition method (ADM) (Adomian, 1994; Adomian and Rach, 1992; Mohammed and Khlaif, 2014; Ogunfiditimi, 2015), the variation iteration method (VIM) (Matinfar *et al.*, 2011; Noor and Mohyud-Din, 2009), the homotopy perturbation method (HPM) (Jafari *et al*,. 2008; He, 2008), etc.

In this paper, the Elzaki transform decomposition method is implemented in the approximation of the analytic solution of the gas dynamic equations. The method merges the properties of the Elzaki transform method (Ziane and Cherif, 2015; Elzaki and Elzaki, 2011a and 2011b; Elzaki, 2012) and the Adomian decomposition method. The method replaces the nonlinear terms with the Adomian polynomials, and writes the solution as the partial sum of the resulting iterates. The method converges favourably to the analytic solution. The method is explicit and reliable as it avoids perturbation, linearization or discretization.

Methods

Preliminaries

i. Let $u(t)$for $t \geq 0$, then the Elzaki transform (Elzaki and Elzaki, 2011a and 2011b) of $u(t)$ (denoted by $E[u]$) is a function of s defined by

$$E[u(t)] = s\int_0^\infty u(t)e^{-\frac{s}{t}}dt.$$

ii. The Elzaki transform of partial derivative is obtained by integration by part as shown in Ziane and Cherif (2015). That is,

$$E[u_t(x,t)] = \frac{1}{r}D(x,r) - ru(x,0),$$

$$E[u_{tt}(x,t)] = \frac{1}{r^2}D(x,r) - ru_t(x,0) - u(x,0),$$

where

$$D_n(x,r) = \frac{D(x,r)}{r^n} - \sum_{k=0}^{n-1} r^{2-n+k}u^{(k)}(0).$$

iii. Some of the Elzaki transform properties can be found in Elzaki and Ezaki (2011b) and Ziane and Cherif (2015), and are given as;

a. $E[1] = r^2$

b. $E[t^n] = n!\, r^{n+2}$

c. $E^{-1}[r^{n+2}] = \frac{t^n}{n!}$.

The above properties will be beneficial in applying the Elzaki transform decomposition method for the Gas dynamic equation in sections 2.4.

iv. To aid our computations, we highlight few standard Elzaki transform for some special functions found in Elzaki (2012).

Table 1: Special functions versus Elzaki transform equivalent

Special Functions	Elzaki Transforms
$u(t)$	$E[u(t)] = D(r)$
$\frac{t^{\alpha-1}}{\Gamma(\alpha)}, \alpha > 0$	$r^{\alpha+1}$
$\frac{t^{n-1}e^{\alpha t}}{(n-1)!}, n = 1,2,\ldots$	$\frac{r^{n+1}}{(1-\alpha r)^n}$
$sin\alpha t$	$\frac{\alpha u^3}{1+\alpha^2 r^2}$
$cos\alpha t$	$\frac{r^2}{1+\alpha^2 r^2}$
$\sinh \alpha t$	$\frac{\alpha r^3}{1-\alpha^2 r^2}$
$\cosh \alpha t$	$\frac{\alpha r^2}{1-\alpha^2 r^2}$
$e^{\alpha t}$	$-\frac{r^2}{1-\alpha r}$

The Adomian Decomposition Method

Consider the standard operation

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \quad (1)$$

with given prescribed auxiliary conditions, L is the highest order derivative which is assumed to be invertible, R is the linear term of order less than L, $Nu(x,t)$ is the nonlinear term, and $f(x,t)$ is the source term (Adomian, 1994; Adomian and Rach, 1992; Mohammed and Khlaif, 2014; Ogunfiditimi, 2015).

Applying the inverse operator L^{-1} to both sides of equation (1), and using the prescribed conditions, we obtain,

$$u(x,t) = L^{-1}\big(f(x,t)\big) - L^{-1}\big(Ru(x,t)\big) - L^{-1}\big(Nu(x,t)\big), \quad (2)$$

The standard Adomian defines the solution $u(x,t)$ as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (3)$$

and the nonlinear term as

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n, \quad (4)$$

where A_n are the Adomian polynomials determined from the relation

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{n} \lambda^i y_i(x,t)\right)\right]_{\lambda=0} \quad (5)$$

If we assumed the nonlinear term $Nu(x,t) = F(u(x,t)$ then the Adomian polynomials are given as

$$A_0 = F\big(u_0(x,t)\big),$$

$$A_1 = u_1(x,t)F\big(u_0(x,t)\big),$$

$$A_2 = u_2(x,t)F'\big(u_0(x,t)\big) + \frac{u_1^2(x,t)}{2!}F''\big(u_0(x,t)\big), \quad (6)$$

$$A_3 = u_3(x,t)F'\big(u_0(x,t)\big) + u_1(x,t)u_2(x,t)F''\big(u_0(x,t)\big) + \frac{u_1^3(x,t)}{3!}F'''\big(u_0(x,t)\big),$$

$$\vdots$$

Elzaki Transform Method

Let us consider the nonlinear PDE of the form

$$\frac{\partial^k u(x,t)}{\partial t^k} + Ru(x,t) + Nu(x,t) = f(x,t), \; k = 1,2,3,4,\dots, \quad (7)$$

with initial condition

$$\left.\frac{\partial^{k-1}u(x,t)}{\partial t^{k-1}}\right|_{t=0} = g_{k-1}, \quad k = 1,2,3,4,\cdots$$

as in Ziane and Cherif (2015), where $\frac{\partial^k u(x,t)}{\partial t^k}$ is the partial derivative of $u(x,t)$ of order k, and is assumed to be invertible, $Nu(x,t)$ is the nonlinear term, R is a linear operator and $f(x,t)$ is the source term.

Applying the Elzaki transforms (see Elzaki and Elzaki, 2011a and 2011b), we obtain

$$E\left[\frac{\partial^k u(x,t)}{\partial t^k}\right] + E[Ru(x,t) + Nu(x,t)] = E[f(x,t)] \quad (8)$$

By preliminary (ii) we have

$$E[u(x,t)] = \sum_{n=0}^{k-1} r^{2+n}\frac{\partial^n u(x,0)}{\partial t^n} + r^k E[f(x,t)] - r^k E[Ru(x,t) + Nu(x,t)] \quad (9)$$

Applying the Elzaki inverse operator, E^{-1}on both sides of (9) we obtain

$$u(x,t) = E^{-1}\left[\sum_{n=0}^{k-1} r^{2+n}\frac{\partial^n u(x,0)}{\partial t^n}\right] + E^{-1}\left[r^k E[f(x,t)]\right] - E^{-1}\left[r^k E[Ru(x,t) + Nu(x,t)]\right], \quad (10)$$

where

$$\left.\frac{\partial^n u(x,0)}{\partial t^n}\right|_{t=0}$$

is the partial derivative of the initial condition.

Elzaki Transform Decomposition Method (ETDM)

In this section, we will make an elegant mixture of the Elzaki transform and the Adomian decomposition method to be tagged the Elzaki transform decomposition method. The method requires that the linear and nonlinear terms in (11) are replaced with

$$\sum_{n=0}^{\infty} u_n(x,t)$$

and

$$\sum_{n=0}^{\infty} A_n(x,t),$$

respectively; where A_n are the Adomian polynomials which are determined from using equation (6), and $u_n(x,t)$ are the components $u_0, u_1, u_2, u_3, \ldots$,which are determined recursively. Hence, equation (10) can be written as

$$\sum_{n=0}^{\infty} u_n(x,t) = E^{-1}\left[\sum_{n=0}^{k-1} r^{2+n}\frac{\partial^n u(x,0)}{\partial t^n}\right] + E^{-1}\left[r^k E[f(x,t)]\right]$$

$$+ \sum_{n=0}^{\infty} A_n(x,t)\Bigg]\Bigg] \quad (11)$$

Comparing both sides of equation (11), we obtain

$$u_0(x,t) = E^{-1}\left[\sum_{n=0}^{k-1} r^{2+n}\frac{\partial^n u(x,0)}{\partial t^n}\right] + E^{-1}\left[r^k E[f(x,t)]\right]$$

$$u_1(x,t) = -E^{-1}\left[r^k E[Ru_0(x,t) + A_0(x,t)]\right]$$

$$u_1(x,t) = -E^{-1}\left[r^k E[Ru_1(x,t) + A_1(x,t)]\right]$$

$$\vdots$$

$$u_{n+1}(x,t) = -E^{-1}\left[r^k E[Ru_n(x,t) + A_n(x,t)]\right], k = 1,2,3,\cdots, n \geq 0 \quad (12)$$

Thus, the approximate solution can be written as

$$u(x,t) = \sum_{n=0}^{N} u_n(x,t) \quad (13)$$

$$- E^{-1}\left[r^k E\left[R\sum_{n=0}^{\infty} u_n(x,t)\right.\right.$$

as $N \to \infty$.

Results (Numerical Applications)

In this section, the Elzaki transform decomposition method is applied to solve homogeneous and non-homogeneous gas dynamic equation. Numerical results are compared with the homotopy perturbation method (HPM) in Jafari *et al.*, (2008) for same problem.

Homogeneous gas Dynamic equation (Jafari *et al.*, 2008)**:**
Consider the given equation

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial (u^2)}{\partial x}-u(1-u)=0,\ 0\leq x\leq 1,\ t>0, \quad (14)$$

with initial condition

$$u(x,0)=e^{-x}.$$

The exact solution is

$$u(x)=e^{t-x}.$$

Applying the Elzaki transform on both sides, we have

$$E\left[\frac{\partial u(x,t)}{\partial t}\right]+E\left[-u(x,t)+\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x}+u^2(x,t)\right)\right]=0$$

By preliminary (ii), we have

$$E[u(x,t)]=r^2u(x,0)-rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x}+u^2(x,t)\right)-u(x,t)\right] \quad (15)$$

Applying the Elzaki inverse operator, E^{-1} on both sides of (15) to obtain

$$u(x,t)=E^{-1}[r^2e^{-x}]-E^{-1}\left[rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x}+u^2(x,t)\right)-u(x,t)\right]\right] \quad (16)$$

By preliminary (iiic), we have

$$E^{-1}[r^2e^{-x}]=e^{-x}E^{-1}[r^2].$$

Hence,

$$u(x,t)=e^{-x}-E^{-1}\left[rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x}+u^2(x,t)\right)-u(x,t)\right]\right] \quad (17)$$

By the Elzaki transform decomposition method, equation (17) can be written as

$$u_0(x,t)=e^{-x},$$

$$u_{n+1}(x,t)=-E^{-1}\left[rE[A_n(x,t)+u_n(x,t)]\right],\ n\geq 0. \quad (18)$$

Hence, for $n=0$, we first compute A_0 using the algorithm in equation (6). Thus,

$$A_0=\frac{1}{2}\frac{\partial(u_0^2)}{\partial x}+u_0^2=0$$

This implies that

$$u_1(x,t)=-E^{-1}\left[rE[-e^{-x}]\right]=te^{-x}$$

For $n=1$,

$$A_1=\frac{d}{dx}\left(\frac{1}{2}\frac{\partial(u_0^2)}{\partial x}+u_0^2\right)u_1=0.$$

Thus

$$u_2(x,t)=-E^{-1}\left[rE[-e^{-x}t]\right]=e^{-x}E^{-1}[r^4]=\frac{t^2}{2!}e^{-x}$$

For $n=2$, $A_2=0$. Hence

$$u_3(x,t)=-E^{-1}\left[rE\left[\frac{t^2}{2!}e^{-x}\right]\right]=e^{-x}E^{-1}[r^5]=\frac{t^3}{3!}e^{-x}.$$

Continuing the above for $n\geq 3$, we obtain

$$u(x,t) = e^{-x} + te^{-x} + \frac{1}{2!}e^{-x}t^2 + \frac{1}{3!}e^{-x}t^3 + \frac{1}{4!}e^{-x}t^4 + \cdots$$
$$= \left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots\right)e^{-x} = e^{t-x} \quad (19)$$

It is obvious that for $n \geq 0$, the approximate solution converges rapidly to the analytic solution. This same result was equally obtained in Jafari *et al.*, (2008) using the method of homotopy perturbation.

Inhomogeneous gas Dynamic equation (Jafari *et al.*, 2008)

Given the equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial(u^2)}{\partial x} - u(1-u) = -e^{(t-x)}, \ \ 0 \leq x \leq 1, \ \ t > 0, \quad (20)$$

with initial condition

$$u(x,0) = 1 - e^{-x}.$$

The exact solution is

$$u(x) = 1 - e^{(t-x)}.$$

Applying the Elzaki transform on both sides, we have

$$E\left[\frac{\partial u(x,t)}{\partial t}\right] + E\left[-u(x,t) + \left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x} + u^2(x,t)\right)\right] = -E[e^{t-x}]$$

By preliminary (ii), we have

$$E[u(x,t)] = r^2u(x,0) - rE[e^{t-x}] - rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x} + u^2(x,t)\right) - u(x,t)\right] \quad (21)$$

Applying the Elzaki inverse operator, E^{-1}on both sides of (21) we obtain

$$u(x,t) = E^{-1}[r^2(1-e^{-x})] - E^{-1}\left[rE[e^{t-x}]\right] - E^{-1}\left[rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x} + u^2(x,t)\right) - u(x,t)\right]\right] \quad (22)$$

$$u(x,t) = E^{-1}[r^2(1-e^{-x})] - E^{-1}\left[re^{-x}E[e^{t}]\right] - E^{-1}\left[rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x} + u^2(x,t)\right) - u(x,t)\right]\right] \quad (23)$$

By preliminary (iv), we have

$$E[e^t] = \frac{r^2}{1-r}.$$

Hence,

$$u(x,t) = E^{-1}[r^2(1-e^{-x})] - E^{-1}\left[re^{-x}\left(\frac{r^2}{1-r}\right)\right] - E^{-1}\left[rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x} + u^2(x,t)\right) - u(x,t)\right]\right]$$

$$u(x,t) = E^{-1}[r^2(1-e^{-x})] + e^{-x}E^{-1}\left[\left(\frac{r^3}{1-r}\right)\right] - E^{-1}\left[rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x} + u^2(x,t)\right) - u(x,t)\right]\right] \quad (24)$$

But,

$$E^{-1}\left[\frac{r^3}{1-r}\right] = 1 - e^t.$$

Therefore, equation (24) becomes

$$u(x,t) = (1-e^{-x}) + e^{-x}(1-e^{-t}) - E^{-1}\left[rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x} + u^2(x,t)\right) - u(x,t)\right]\right]$$

$$= 1 - e^{t-x} - E^{-1}\left[rE\left[\left(\frac{1}{2}\frac{\partial\left(u^2(x,t)\right)}{\partial x} + u^2(x,t)\right) - u(x,t)\right]\right] \quad (25)$$

By the Elzaki transform decomposition method, equation (25) can be written as

$$u(x,0) = 1 - e^{t-x},$$

$$u_{n+1}(x,t) = -E^{-1}\left[rE[A_n(x,t) - u_n(x,t)]\right], n \geq 0. \quad (26)$$

Hence, for $n = 0$, we first compute A_0 using the algorithm given in equation (6).Thus,

$$A_0 = \frac{1}{2}\frac{\partial(u_0^2)}{\partial x} + u_0^2 = 0.$$

Thus

$$u_1(x,t) = 0.$$

.For $n = 1$, we also have that

$$u_2(x,t) = 0.$$

Continuing the above process, we have

$$u_{n+1}(x,t) = 0, n \geq 0.$$

Hence,

$$u(x,t) = 1 - e^{t-x}. \quad (27)$$

Here, the initial approximate solution gives the analytic solution. This same result was equally obtained in Jafari *et al.*, (2008) using the method of homotopy perturbation for $n \geq 0$.

Discussion of Results

The Elzaki transform decomposition method has been successively applied to finding the exact solution of analytic solution of homogeneous and non-homogeneous gas dynamic equations with less computation effort. From the result obtained, we can authoritatively accent that the Elzaki transform decomposition method converges favourably to the analytic solution. It gives the solution as a convergent series which converges rapidly to the exact solution. The method requires no additional cost such as perturbation, linearization or discretization.

Conclusion

The method has been successively implemented for obtaining the analytic solution of gas dynamic equations. The method, which merges the properties of the Elzaki Transform method and Adomian decomposition method proved highly effective in resolving the gas dynamic equations. This method can be further explored to solve problems such as stochastic partial differential equations, nonlinear delay dynamic systems, delay differential equations, etc.

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