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Certainty Equivalence Actuarial Risk Aversion

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Abstract
Issues in utility theory ranking and economics of insurance, involve certainty equivalent value and Arrow Pratt aversion as the main target of choosing between different alternative investment options and a metric for cost-effective investment decisions. The conventional certainty method addresses the ranking of investment with a desired high level precision. Further, conditions for consistent approximations of certainty equivalence and risk aversion of interest by Taylor's method are presented. The presentation is direct so as to give room for easy understanding of the theme justifying that Taylor's scheme offers an edge over other methods. In this paper an analytic procedure used in calculating certainty equivalence employing the tool of risk aversion is considered by subjecting the minimum premium of insurer to analyticity testing and then solve the equation $E(U(w+\sum^{-}-Y))=w$ for the minimum premium $\sum^{-}$ the insurer is ready to impose on the insured. The resulting solution is found to be integral of the form $\sum^{-}=\int_{-\infty}^{\infty} y f_Y(y)dy$.

Keywords: Aversion, utility, expectation, certainty equivalence

1. Introduction

In mathematics of insurance market, the expected utility criterion is usually employed to order investment alternative decision but it does not have a very sound actuarial basis for using it as a ranking scale. It is just a real number which assumes a value hence there is a need for an alternative ranking method founded on probability distribution basis having power of certainty value. In Fishburn (1989); Fishburn and Walker P (1995), it was apparent that the expected utility theory is a mathematical tool which the insured uses to choose between two risky insured investments by contrasting the expected utilities and this is what prompted Davis; Hands; Maki; London; Elgar (1987) to define expected utility as the product of the respective probability values and the weighted sums derived by summing the utilities over all outcomes. The insurance underwriting is a very complex procedure of combining underwriting and investment decisions the performance of which is critical to all stakeholders and hence risk managers must be decisive enough before earned premium income is invested. The efficient ranking of different investments is a critical problem in insurance and financial applications which must be addressed within actuarial framework. The methodology of expected utility theory which does not provide sound actuarial basis for justification informs the decision of actuarial risk experts to dig for an alternative technique which employs certainty using probability distribution defined over the outcomes and utilities as the most appropriate. Friedmann and Savage (1952) both keenly observe that provided that if policy holders realize a few elementary rules of behaviour such as certainty equivalence, then risk preferences can be expressed rationally. The theory of certainty equivalence scheme in principle is not only applicable to financial models but can be extended to critical areas of actuarial literature.

In Carol (2008), McCutcheon and Scott (1986), Wilmott (2007) one can observe which insurance portfolio available to an insurer is regarded as optimal apparently relies on the perceived attitude to risk and trade-off between risks and return, hence an insured that is very averse to taking risks simply suggests that he is more likely to choose a low return and low risk portfolio over a high risk and high return portfolio. To formulate the choice of optimal insurance portfolio in an actuarial framework and find a unique mathematical solution, it is pertinent that the preferences of an investor be defined in terms of utility function the phenomenon which aptly describes an investor's attitude to risk. For an insurer to stay competitive and hedge against total failure, the need for thorough and careful examination from a mathematical point of view is fundamental. An analytical model formulation is of considerable value but with more complicated models which are increasingly involved and technically demanding, the search for an analytical answer to premium pricing and hedging problems could be extremely challenging and the only possibility is to resort to numerical procedures. This paper is anchored on numerical methods applied to problems in mathematics of risk theory.

2. Theoretical Background and Applications

This paper covers critical areas in insurance risk mathematics such as certainty equivalence, risk aversion. The numerical methods are varying; differential co-efficient, Taylor's theorem and numerical limit theorems are all used where applicable. While the expected utility can be used to order investment alternatives, it is not as good ranking measurement scale as certainty equivalence. To start with, there is need to compute the certainty equivalence function defined by $U(c) = E(U(X))$. There exists a certain equivalent function such that an investor is indifferent between uncertain and a certain equivalent value. An investment I may be described by a probability distribution function over a set of associated outcomes or utilities with all possible outcomes so that if $u(w_j)$, j = 1,2,3,…,n be a sequence of utility functions having probabilities p_j, so that $E[u(w)] = \sum_1^n u(w_j)p_j$. I describes an investment whose utility function has been given as $u(w) = f(w)$ and the distribution is further defined by $u(C_e I) = E(U(I)) = \sum_1^n u(w_j)p_j$. We consider the distribution given below for N investments.

Table 1: Probability Distribution for N Investments

$\mathbf{U(w)}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$	.	$\mathbf{w_{n-2}}$	$\mathbf{w_{n-1}}$	$\mathbf{w_n}$
Pr(investmentG_1)	α_{11}	α_{12}	α_{13}	α_{14}	.	$\alpha_{1(n-2)}$	$\alpha_{1(n-1)}$	α_{1n}
Pr(investmentG_2)	α_{21}	α_{22}	α_{23}	α_{24}	.	$\alpha_{2(n-2)}$	$\alpha_{2(n-1)}$	α_{2n}
Pr(investmentG_3)	α_{31}	α_{32}	α_{33}	α_{34}	.	$\alpha_{3(n-2)}$	$\alpha_{3(n-1)}$	α_{3n}
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.								
Pr(investment $G_{(N-1)}$)	$\alpha_{(N-1)1}$	$\alpha_{(N-1)2}$	$\alpha_{(N-1)3}$	$\alpha_{(N-1)4}$	.	$\alpha_{(N-1)(n-2)}$	$\alpha_{(N-1)(n-1)}$	$\alpha_{(N-1)n}$
Pr(investmentG_N)	α_{N1}	α_{N2}	α_{N3}	α_{N4}	.	$\alpha_{N(n-2)}$	$\alpha_{N(n-1)}$	α_{Nn}

$$\sum_{j=1}^{n}\alpha_{1j}=\sum_{j=1}^{n}\alpha_{2j}=\sum_{j=1}^{n}\alpha_{3j}=\ldots=\sum_{j=1}^{n}\alpha_{nj}=1$$

For $\alpha_{nj}\neq\alpha_{mj}$

$$E(U(G_1))=w_1\alpha_{11}+w_2\alpha_{12}+w_3\alpha_{13}+\ldots+w_n\alpha_{1n}$$
$$E(U(G_2))=w_1\alpha_{21}+w_2\alpha_{22}+w_3\alpha_{23}+\ldots+w_n\alpha_{2n}$$
$$E(U(G_3))=w_1\alpha_{31}+w_2\alpha_{32}+w_3\alpha_{33}+\ldots+w_n\alpha_{3n}$$

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$$E(U(G_N))=w_1\alpha_{N1}+w_2\alpha_{N2}+w_3\alpha_{N3}+\ldots+w_n\alpha_{Nn}$$

Since $u(w)=f(w)$, then $u(c_e)=f(c_e)$

$f(c_eG_1)=w_1\alpha_{11}+w_2\alpha_{12}+w_3\alpha_{13}+\ldots+w_n\alpha_{1n}$ solving for certainty equivalence, we have

$c_eG_1=f^{-1}(w_1\alpha_{11}+w_2\alpha_{12}+w_3\alpha_{13}+\ldots+w_n\alpha_{1n})$ certainty equivalence for G_1

$$f(c_eG_k)=w_1\alpha_{k1}+w_2\alpha_{k2}+w_3\alpha_{k3}+\ldots+w_n\alpha_{kn}$$
$$c_eG_k=f^{-1}(w_1\alpha_{k1}+w_2\alpha_{k2}+w_3\alpha_{k3}+\ldots+w_n\alpha_{kn})$$

If $f^{-1}(w_1\alpha_{11}+w_2\alpha_{12}+w_3\alpha_{13}+\ldots+w_n\alpha_{1n})\succ f^{-1}(w_1\alpha_{k1}+w_2\alpha_{k2}+w_3\alpha_{k3}+\ldots+w_n\alpha_{kn})$ implies G_1 is preferred to G_k, otherwise G_k is preferred. Often times, we determine preference by computing $\max c_eG_k=\max f^{-1}(w_1\alpha_{k1}+w_2\alpha_{k2}+w_3\alpha_{k3}+\ldots+w_n\alpha_{kn})$. However, when equality occurs, then $f^{-1}(w_1\alpha_{11}+w_2\alpha_{12}+w_3\alpha_{13}+\ldots+w_n\alpha_{1n})=f^{-1}(w_1\alpha_{k1}+w_2\alpha_{k2}+w_3\alpha_{k3}+\ldots+w_n\alpha_{kn})$ preference goes to either of them. Although, both the certainty equivalence and expected utility are numbers that are used in ranking investments, certainty equivalence is better since it is based on probability making it more intuitive in interpretation than expected utility

A utility function $u: P \to R$ is the expected utility if $\exists$, utility values $\{u_1, u_2, u_3,..., u_n\}$ for each of the n outcomes in $\{x_1, x_2, x_3,..., x_n\}$such that for every p in P, $U(P) = \sum_{i=1}^{n} p_i u_i$

It can be inferred from the properties of utility function that $U(P)$ is linear in probability $u(\delta p_1 + (1-\delta)p_2) = \delta U(p_1) + (1-\delta)U(p_2)$. If an insured decision to insure or not is described by the expected utility function, then the insured payoff can be underpinned over uncertain outcome $\{x_1, x_2, x_3,..., x_n\}$.

2.1 Theorem 1

If $U(P)$ is linear in probability and $u(\delta p_1 + (1-\delta)p_2) = \delta U(p_1) + (1-\delta)U(p_2)$ holds, then it has expected utility functional form.

Proof

The condition $u(\delta p_1 + (1-\delta)p_2) = \delta U(p_1) + (1-\delta)U(p_2)$ confirms that U(.) is a linear functional

$$\delta U(p_1) + (1-\delta)U(p_2) = \delta\sum_{i=1}^{n} p_{1i}u_i + (1-\delta)\sum_{i=1}^{n} p_{2i}u_i$$

$$\delta U(p_1) + (1-\delta)U(p_2) = \sum_{i=1}^{n} [\delta p_{1i}u_i + (1-\delta)p_{2i}u_i]$$

Since $0 \le \delta \le 1$, choose $\delta = 0.5$, $0.5U(p_1) + 0.5U(p_2) = \sum_{i=1}^{n} [0.5p_{1i}u_i + 0.5p_{2i}u_i]$

$$0.5U(p_1) + 0.5U(p_2) = 0.5\sum_{i=1}^{n} [p_{1i}u_i + p_{2i}u_i]$$

$$U(p_1) + U(p_2) = \sum_{i=1}^{n} [p_{1i} + p_{2i}]u_i = \sum_{i=1}^{n} p_i\, u_i ,.................\text{Result1}$$

$p_i = p_{1i} + p_{2i}$,

hence the proof

Now suppose

$F_X(x) = \Pr(X \le x)$ be the probability of receiving less than or equal to x

$$U(F) = \int_{-\infty}^{\infty} u(x) f_X(x) dx ,$$

If $E(F) = \int_{-\infty}^{\infty} x f_X(x) dx$ is the expected value of the function F(.) and $\psi_{E(F)}$ is the functional which gives $\int_{-\infty}^{\infty} x f_X(x) dx$ for certain, then the insured will prefer $\psi_{E(F)}$ to F(.). Therefore $\int_{-\infty}^{\infty} u(x) f_X(x) dx \le u\left(\int_{-\infty}^{\infty} x f_X(x) dx\right) = U(E(F))$ would then be the risk aversion for all $f_X(x)$.

Abouda and Farhoud (2010) define risk aversion as an attempt to choose between alternatives when clearly feasible so as to avoid risk. According to the authors a weak risk averse potential policy holder would prefer a random risk Y whose certainty of the expected value is E(Y) however in Rothschild and Stiglitz (1970), it was observed that a strong risk aversion will describe an aversion towards the average preserving spreads. Machina (1982) also is of the opinion that a policy holder with everywhere concave local utility functional should meet the conditions of the preference ordering of second order stochastic dominance but would be strictly worse off due to the inclusion of independent underlying risk. To Quiggin ((2003), however, in a wide range of risk averse preferences generated from other models under uncertainty, independent risks are complementary where aversion to a risk will diminish under the presence of an independent underlying risk hence reactions of a risk averse policy holder with expected utility preference to the inclusion of independent underlying risk makes it a critical area which requiring serious attention. A possible consequence of the author's work falling in line with the concept of risk aversion is that the inclusion of independent risk will decline welfare and that additional risk will be worse off.

II Certainty Equivalence of Risk Aversion

Let X define the insured risk, then it becomes a random variable, For every X, there is a value $c = c(X)$, the certainty equivalence such that $c \approx X$ where a policy holder is indifferent to either insure or not. From the point of view of the insured, the $u(x)$ is concave in an interval O if for the numbers x_1, x_2 in O and for $0 \le \alpha \le 1$

$$\alpha u(x_1) + (1-\alpha)u(x_2) \le u(\alpha x_1 + (1-\alpha)x_2).$$

Substitute $\alpha = 0.5$

$$0.5u(x_1) + 0.5u(x_2) \le u(0.5x_1 + 0.5x_2)$$

Rotar (2007) defines aversion as

$$A_\delta(x) = \begin{cases} \delta, \text{Probability} = 0.5 \\ -\delta, \text{probability} = 0.5 \end{cases}$$

number c describes the correct price of X. From above, it should be stated that c is not a random number. Therefore $u(c) = u(X) \Rightarrow E(u(c)) = E(u(X))$ and by the requirement that c is not a random variable $u(c) = E(u(X)) \le u(E(X))$ by Jensen's inequality.

$f_X(x)$ and $f_{A_\delta}(a)$ are the probability densities of random variables X and $A_\delta(x)$. The risk random variables X and aversion $A_\delta(x)$ are assumed independent.

$$E(u(X + A_\delta)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x+a) f_X(x) f_{A_\delta}(a)\,dx\,da = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u(x+a) f_{A_\delta}(a)\,da\right) f_X(x)\,dx$$

$$E(u(X + A_\delta)) = \int_{-\infty}^{\infty} 0.5u(x+\delta) + 0.5u(x-\delta) f_X(x)\,dx = E(0.5u(x+\delta) + 0.5u(x-\delta))$$

$E(u(X+A_\delta)) \le E(u(X))$ and hence $X + A_\delta \approx X$ It is apparent here that the increase in inequality associated with a given source of inequality is reduced in the presence of other statistically independent source of inequality.

Proof

$$U(x) = U(\mu_x) + U^{(1)}(x)(x-\mu_x) + \frac{U^{(2)}(x)(x-\mu_x)^2}{2!} + \frac{U^{(3)}(x)(x-\mu_x)^3}{3!} + \frac{U^{(4)}(x)(x-\mu_x)^4}{4!}$$

$$+ \ldots + \frac{U^{(n-1)}(x)(x-\mu_x)^{n-1}}{(n-1)!} + \frac{U^{(n)}(x)(x-\mu_x)^n}{n!}, |X - E(X)|^k \to o(1) \text{ for } k$$

≥ 3 where $\mathbf{o}(1)$ is function which vanishes.

$$U(x) \cong U(\mu_x) + U^{(1)}(x)(x-\mu_x) + \frac{U^{(2)}(x)(x-\mu_x)^2}{2!} + 0 + 0 + \ldots$$

$$U(x) \cong U(\mu_x) + U^{(1)}(x)(x-\mu_x) + \frac{U^{(2)}(x)(x-\mu_x)^2}{2!}$$

$$U(c) \cong U(\mu_x) + \frac{\left(U^{(2)}(x)E(x-\mu_x)^2\right)}{2!}$$

$$U(c) \cong U(\mu_x) + 0.5U^{(2)}(x)\sigma^2{}_x$$

The aversion in terms of second and first derivatives is defined as $a(x) = -\frac{U^{(2)}(x)}{U^{(1)}(x)}$

While the relative aversion is $r(x) = -\frac{xU^{(2)}(x)}{U^{(1)}(x)} \Rightarrow U^{(2)}(x) = -\frac{U^{(1)}(x)r(x)}{x}$

$$U^{(2)}(E(x)) = -U^{(1)}(E(x))a(E(x))$$

$$U^{(2)}(\mu_x) = -U^{(1)}(\mu_x)a(\mu_x)$$

2.2 Theorem 2

Suppose X is a random variable, then $U(\mu_x) - 0.5\,(C.V)^2 \mu_x{}^2 U'(\mu_x)\, a(\mu_x) \le U(E(X))$ where C.V is the co-efficient of variation, where $a(\mu_x)$ is the aversion coefficient. $(u(c)) = E(u(X))$

$E(x) = \mu_x$, $\text{var}(x) = \sigma^2{}_x$

Taking mathematical expectation of both sides, we have

$$EU(x) \cong EU(\mu_x) + E\left(U^{(1)}(x)(x-\mu_x)\right) + \frac{E\left(U^{(2)}(x)(x-\mu_x)^2\right)}{2!}$$

$$U(x) \cong U(\mu_x) + \left(U^{(1)}(x)E(x-\mu_x)\right) + \frac{\left(U^{(2)}(x)E(x-\mu_x)^2\right)}{2!}$$

$$U(x) \cong U(\mu_x) + \left(U^{(1)}(x)(Ex-\mu_x)\right) + \frac{\left(U^{(2)}(x)E(x-\mu_x)^2\right)}{2!}$$

$$U(x) \cong U(\mu_x) + \left(U^{(1)}(x)(\mu_x-\mu_x)\right) + \frac{\left(U^{(2)}(x)E(x-\mu_x)^2\right)}{2!}$$

$$\Rightarrow U^{(2)}(x) = -U^{(1)}(x)a(x)$$

By substituting $E(x)$ into $U^{(2)}(x) = -U^{(1)}(x)a(x)$ we have

$$U(c) \cong U(\mu_x) - 0.5U^{(1)}(\mu_x)a(\mu_x)\sigma^2{}_x$$

The co-efficient of variation $CV = \frac{\sigma_x}{\mu_x}$

$$U(c) \cong U(\mu_x) - 0.5U^{(1)}(\mu_x)a(\mu_x)(CV)^2 \mu^2{}_x$$

Recall that $u(c) = E(u(X))$ and by Jensen's inequality $E(u(X)) \le u(E(X))$, hence

$$U(\mu_x) - 0.5U^{(1)}(\mu_x)a(\mu_x)(CV)^2 \mu_x{}^2 \le E(x), \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{Result 2}$$

This result is of high degree of interest with respect to inequality where analyticity testing is finally anchored on Jensen's inequality displaying the homothetic property of aversion risk.

III Minimum Premium for any Risk

From the theoretical point of view, insurance premium can be viewed from two opposing sides. The insurer will attempt to charge the minimum premium while the policy holder is comfortable with the maximum amount he can afford to pay. For any risk Y while ignoring additional costs, a premium E[Y] will be sufficient.

2.3 Theorem 3

The solution to equation $E(U(w + \Sigma^- - Y)) = w$ is integral and defines the minimum mean the insurer imposes on the insured

Proof

$$U(w - y + \Sigma^-) = u(w - y) + \Sigma^- u^{(1)}(w - y) + \frac{(\Sigma^-)^2 u^{(2)}(w - y)}{2!}, \text{or}$$

$$aU(w - y + \Sigma^-) + b = au(w - \mu_y) + au'(w - \mu_y)(\mu_y - y)^1 + \frac{au''(w - \mu_y)(\mu_y - y)^2}{2!} +$$

$$EU(w - y + \Sigma^-) = w \quad a(\Sigma^-)u'(w - \mu_y) + a(\Sigma^-)u''(w - \mu_y)(\mu_y - y) + \frac{a(\Sigma^-)u'''(w - \mu_y)(\mu_y - y)^2}{2!}$$

$+$

$$0.5a(\Sigma^{-1})^2 u''(w - \mu_y) + 0.5a(\Sigma^{-1})^2 u'''(w - \mu_y)(\mu_y - y) + \frac{a(\Sigma^{-1})^2 u''''(w - \mu_y)(\mu_y - y)^2}{4} + b$$

we take mathematical expectation of left hand side alone to obtain,

$$E\left\{\begin{array}{l} au(w - \mu_y) + au'(w - \mu_y)(\mu_y - y)^1 + \frac{au''(w - \mu_y)(\mu_y - y)^2}{2!} + \\ a(\Sigma^-)u'(w - \mu_y) + a(\Sigma^-)u''(w - \mu_y)(\mu_y - y) + \frac{a(\Sigma^-)u'''(w - \mu_y)(\mu_y - y)^2}{2!} \\ + 0.5a(\Sigma^{-1})^2 u''(w - \mu_y) + 0.5a(\Sigma^{-1})^2 u'''(w - \mu_y)(\mu_y - y) + \frac{a(\Sigma^{-1})^2 u''''(w - \mu_y)(\mu_y - y)^2}{4} + b = aw + b \end{array}\right\}$$

$$aEu(w-\mu_y)+au'(w-\mu_y)(\mu_y-Ey)^1+\frac{au''(w-\mu_y)E(\mu_y-y)^2}{2!}+$$
$$a(\Sigma^-)Eu'(w-\mu_y)+a(\Sigma^-)u''(w-\mu_y)(\mu_y-Ey)+\frac{a(\Sigma^-)u'''(w-\mu_y)E(\mu_y-y)^2}{2!}$$
$$+0.5a(\Sigma^{-1})^2Eu''(w-\mu_y)+0.5a(\Sigma^-)^2u'''(w-\mu_y)(\mu_y-Ey)+\frac{a(\Sigma^{-1})^2u''''(w-\mu_y)E(\mu_y-y)^2}{4}+b=aw+b$$

$$aEu(w-\mu_y)+au'(w-\mu_y)(\mu_y-\mu_y)^1+\frac{au''(w-\mu_y)\sigma^2{}_y}{2!}+$$
$$a(\Sigma^-)Eu'(w-\mu_y)+a(\Sigma^-)u''(w-\mu_y)(\mu_y-\mu_y)+\frac{a(\Sigma^-)u'''(w-\mu_y)\sigma^2{}_y}{2!}$$
$$+0.5a(\Sigma^{-1})^2Eu''(w-\mu_y)+0.5a(\Sigma^-)^2u'''(w-\mu_y)(\mu_y-\mu_y)+\frac{a(\Sigma^{-1})^2u''''(w-\mu_y)\sigma^2{}_y}{4}+b=aw+b$$

$$aEu(w-\mu_y)+\frac{au''(w-\mu_y)\sigma^2{}_y}{2!}+a(\Sigma^-)Eu'(w-\mu_y)+\frac{a(\Sigma^-)u'''(w-\mu_y)\sigma^2{}_y}{2!}$$
$$+0.5a(\Sigma^{-1})^2Eu''(w-\mu_y)+\frac{a(\Sigma^{-1})^2u''''(w-\mu_y)\sigma^2{}_y}{4}+b=aw+b$$

$$Eu(w-\mu_y)+\frac{u''(w-\mu_y)\sigma^2{}_y}{2!}+(\Sigma^-)Eu'(w-\mu_y)+\frac{(\Sigma^-)u'''(w-\mu_y)\sigma^2{}_y}{2!}$$
$$+0.5(\Sigma^{-1})^2Eu''(w-\mu_y)+\frac{(\Sigma^{-1})^2u''''(w-\mu_y)\sigma^2{}_y}{4}=w$$

as we let $u^{(k)}(w-\mu_Y)\to \mathbf{o}(1),\ k\geq 2$ since naturally $u^{(k)}(w-\mu_Y)$ all vanish for a linear utility function

$$Eu(w-\mu_y)+(\Sigma^-)Eu'(w-\mu_y)=w$$

$u(w-\mu_Y)+(\Sigma^-)u'(w-\mu_Y)=w$, a has cancelling out.

$u'(w-\mu_Y)=k$ for linear utility function, $u(w-\mu_Y)$ is of the form $bw+a=w-\mu_Y$

$w-\mu_Y+k\Sigma^-=w\Rightarrow$

$-\mu_Y+k\Sigma^-=0$ and the result follows if k =1

$\Sigma^-=\mu_Y\Rightarrow$

$$\Sigma^-=\int_{-\infty}^{\infty} yf_Y(y)dy,\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots\ldots \text{Result 3}$$

is the solution and defines the minimum premium the insurance company will charge the insured. This also defines the mathematical expectation of the distribution for the risk

If $\Sigma^-=E(Y)$ then, $E(w)=\Sigma^-$. Utility function of the form $u(w)=w$ is referred to as risk neutral and an insurer associated with it is a risk neutral investor.

$E[U(w)]=U(c)$; the expectation of a utility function is a constant resulting by equating the expectation to the original utility functional form and solving for c.

IV Aversion Criterion for Standard Utility Functions

Unless we know the mathematical expectation and the variance of random variables Y for quadratic, logarithmic, power and linear transform utility function, we may not be able to compute their certainty equivalents. In particular the first two utility functions discussed possess striking properties.

Exponential Utility Functions

$\mu_Y=\frac{1}{\alpha}$, $\sigma^2{}_y=\frac{1}{\alpha^2}$

$u(w)=-\alpha e^{-\alpha w},\ u(\mu_Y)=-\alpha e^{-\alpha\mu_Y}$

$u'(w)=\alpha^2e^{-\alpha w},\ u'(\mu_Y)=\alpha^2e^{-\alpha\mu_Y}$

$u''(w)=-\alpha^3e^{-\alpha w},\ u''(\mu_Y)=-\alpha^3e^{-\alpha\mu_Y}$

$a(\mu_Y)=\frac{-u''(w)}{u'(w)}=\frac{\alpha^3}{\alpha^2}=\alpha>0$, can re-written as $\alpha=\frac{1}{\beta}$

$U(c)\cong U(\mu_y)-0.5U'(\mu_y)\,a(\mu_x)\sigma^2{}_y$

$U(c) \cong -\alpha e^{-\alpha\mu_Y} + \frac{-0.5\alpha^3 e^{-\alpha\mu_y}}{\alpha^2} = -\alpha e^{-\alpha\mu_Y}$

$-0.5\alpha^3 e^{-\alpha\mu_Y}$, but

$U(c) = -\alpha e^{-\alpha c}$

$-\alpha e^{-\alpha c} = -\alpha e^{-\alpha\mu_Y} - 0.5\alpha^3 e^{-\alpha\mu_Y}$

$e^{-\alpha c} = e^{-\alpha\mu_Y} + 0.5\alpha^2 e^{-\alpha\mu_Y}$

$e^{\alpha\mu_Y - \alpha c} = 1 + 0.5\alpha^2$, thus expanding the left hand side up to second order

$1 + \alpha\mu_Y - \alpha c + 0.5(\alpha\mu_Y - \alpha c)^2 \cong 1 + 0.5\alpha^2$

$1 + \alpha\mu_Y - \alpha c + 0.5(\alpha^2\mu_Y{}^2 + \alpha^2 c^2 - 2\mu_Y\alpha^2 c) \cong 1 + 0.5\alpha^2$

$1 + \alpha\mu_Y - \alpha c + 0.5\alpha^2\mu_Y{}^2 + 0.5\alpha^2 c^2 - \mu_Y\alpha^2 c \cong 1 + 0.5\alpha^2$,

$0.5\alpha^2 c^2 - (\mu_Y\alpha^2 + \alpha)c + \alpha\mu_Y + 0.5\alpha^2\mu_Y{}^2 - 0.5\alpha^2 = 0$

$$c \cong \frac{\left(\mu_Y\alpha^2 + \alpha\right) \pm \sqrt{\left(\mu_Y\alpha^2 + \alpha\right)^2 - 2\alpha^2\left(\mu_Y\alpha + 0.5\mu^2{}_Y\alpha^2 - 0.5\alpha^2\right)}}{\alpha^2}$$

$$c \cong \frac{\left(\mu_Y\alpha^2 + \alpha\right) \pm \sqrt{\alpha^2 + \alpha^4}}{\alpha^2} = \frac{\left(\mu_Y\alpha^2 + \alpha\right) \pm \alpha\sqrt{1+\alpha^2}}{\alpha^2} = \frac{\left(\mu_Y\alpha^2 + \alpha\right) \pm \sqrt{1+\alpha^2}}{\alpha}, \text{.......................Result4}$$

Table 2: Probability Distribution 2 for Risk Y

Y	$\mathbf{Y_1}$	$\mathbf{Y_2}$	$\mathbf{Y_3}$	$\mathbf{Y_4}$	**.**	$\mathbf{Y_{n-2}}$	$\mathbf{Y_{n-1}}$	$\mathbf{Y_n}$
Probability	α_{11}	α_{12}	α_{13}	α_{14}	.	$\alpha_{1(n-2)}$	$\alpha_{1(n-1)}$	α_{1n}

$E(Y) = \alpha_{11}Y_1 + \alpha_{12}Y_2 + \alpha_{13}Y_3 + \alpha_{14}Y_4 + \ldots + \alpha_{1(n-1)}Y_{(n-1)} + \alpha_{1n}Y_n = \sum_1^n \alpha_{1k}Y_k$

$E(Y^2) = \alpha_{11}Y_1{}^2 + \alpha_{12}Y_2{}^2 + \alpha_{13}Y_3{}^2 + \alpha_{14}Y_4{}^2 + \ldots + \alpha_{1(n-1)}Y_{(n-1\}}{}^2 + \alpha_{1n}Y_n{}^2 = \sum_1^n \alpha_{1k} Y_k{}^2$

$\sigma^2{}_Y = E(Y^2) - [E(Y)]^2 = \sum_1^n \alpha_{1k} Y_k{}^2 - (\sum_1^n \alpha_{1k}Y_k{}^2)^2$

$U(c) \cong U(\sum_1^n \alpha_{1k}Y_k) - 0.5U'(\sum_1^n \alpha_{1k}Y_k)\, a(\sum_1^n \alpha_{1k}Y_k) \{\sum_1^n \alpha_{1k} Y_k{}^2 - (\sum_1^n \alpha_{1k}Y_k{}^2)^2\}$

Hara Utility Function

$U(w) = \frac{(w+c)^\alpha}{\alpha}$, $y > -c$ and $0 < \alpha < 1$

$u(\mu_Y) = \frac{(\mu_Y+c)^\alpha}{\alpha}$

$u'(w) = (w+c)^{\alpha-1}$, $u'(\mu_Y) = (\mu_Y + c)^{\alpha-1}$

$u''(w) = (\alpha-1)(w+c)^{\alpha-2}$, $u''(w) = (\alpha-1)(\mu_Y+c)^{\alpha-2}$

$a(w,\mu_Y) = \frac{-u''(w)}{u'(w)} = \frac{-(\alpha-1)(\mu_Y+c)^{\alpha-2}}{(\mu_Y+c)^{\alpha-1}} = -(\alpha-1)(\mu_Y+c)^{-1} = \frac{-(\alpha-1)}{(\mu_Y+c)}$

Property of Hara Utility Function

We cannot absolutely divorce utility function from regular varying functions, in particular, the HARA utility function since a few heavy tailed distributions are the distributions with regularly varying tail distributions.

2.4 Theorem 4

HARA utility function is regularly varying

Proof

By definition the positive, measurable utility function U(w) will be called regularly varying as

$w \to \infty$ with real index α of variation, if $\lim_{w \to \infty} \frac{U(\gamma w)}{u(w)} = \gamma^{\alpha}$ exists for any $\gamma > 0$. If $\alpha = \mathbf{o}(1)$, then $u(w)$ is a slowly varying function.

$U(w) = \frac{(w+c)^{\alpha}}{\alpha}$,

$U(\gamma w) = \frac{(\gamma w+c)^{\alpha}}{\alpha}$

$\frac{U(\gamma w)}{U(w)} = \frac{(\gamma w+c)^{\alpha}}{(w+c)^{\alpha}} = \left\{\frac{(\gamma w+c)}{(w+c)}\right\}^{\alpha} = \left(\gamma + \frac{c-\gamma c}{w+c}\right)^{\alpha}$

Let $Y = \left(\gamma + \frac{c-\gamma c}{w+c}\right)^{\alpha}$

$Y^{\frac{1}{\alpha}} = \left(\gamma + \frac{c-\gamma c}{w+c}\right)$

$\lim_{w \to \infty} Y^{\frac{1}{\alpha}} = \lim_{w \to \infty} \left(\gamma + \frac{c-\gamma c}{w+c}\right)$

$Y^{\frac{1}{\alpha}} = \gamma$

$Y = \gamma^{\alpha}$

$Y = \frac{U(\gamma w)}{U(w)} = \gamma^{\alpha}$

Quadratic Utility Function

$u(w) = -(\alpha - w)^2$, $w < \alpha$ and

$U(w) = 0$ if $w > \alpha$

$u(\mu_Y) = -(\alpha - \mu_Y)^2$,

$u'(w) = 2(\alpha - w)$, $u'(\mu_Y) = 2(\alpha - \mu_Y)$

$u''(w) = -2$, $u''(\mu_Y) = -2$

$a(\mu_Y) = \frac{-u''(\mu_Y)}{u'(\mu_Y)} = \frac{2}{2(\alpha-\mu_Y)} = \frac{1}{(\alpha-\mu_Y)}$

Logarithmic Utility Function

$u(w) = \log_e(\alpha + w)$

$u(\mu_Y) = \log_e(\alpha + \mu_Y)$,

$u'(w) = \frac{1}{(\alpha+w)} = (\alpha + w)^{-1}$, $u'(\mu_Y) = (\alpha + \mu_Y)^{-1}$,

$u''(w) = -(\alpha + w)^{-2}$, $u''(\mu_Y) = -(\alpha + \mu_Y)^{-2}$

$a(w, \mu_Y) = \frac{-u''(w)}{u'(w)} = \frac{(\alpha+\mu_Y)^{-2}}{(\alpha+\mu_Y)^{-1},} = \frac{1}{\alpha+\mu_Y}$, $\alpha > -\mu_Y$

Power Utility Function

$u(w) = w^c = e^{c\log_e w}$ $0 < c < 1$

$u(\mu_Y) = {\mu_Y}^c = e^{c\log_e \mu_Y}$

$u'(w) = \frac{c}{w} e^{c\log_e w} = cw^{c-1}$

$u'(\mu_Y) = \frac{c}{\mu_Y} e^{c\log_e \mu_Y} = c{\mu_Y}^{c-1}$

$u''(w) = c(c-1)w^{c-2}$

$u''(\mu_Y) = c(c-1){\mu_Y}^{c-2}$

$a(w, \mu_Y) = \frac{-u''(w)}{u'(w)} = \frac{c(c-1){\mu_Y}^{c-2}}{c{\mu_Y}^{c-1}} = \frac{(c-1){\mu_Y}^{c-2}}{{\mu_Y}^{c-1}} = (c-1){\mu_Y}^{-1}$

For exponential utility, $a(w) = \alpha$, but one can assume this to be of the form $\frac{1}{\beta}$

For quadratic $a(w) = \frac{1}{(\alpha-\mu_Y)}$

For logarithm $a(w) = \frac{1}{\alpha+\mu_Y}$

For power utility $a(w) = \frac{(c-1)}{\mu_Y}$

A keen observation of the aversion co-efficient of all the standard utility functions shows that they are of the form $a(y) = (a + by)^{-1}$, and taking reciprocals of both sides,

$\frac{1}{a(y)} = a + by \Rightarrow$

$ABRT(y) = a + by$ implying absolute risk tolerance is of linear form and hence represents a risk neutral function.

Conclusion

This paper examines certainty equivalence of risk aversion. An upper bound has been constructed for the certainty $U(c)$ by the conditions of Jensen's inequality where the aversion co-efficient has been embedded in the presence of co-efficient of variation. To test the efficacy of the certainty approximation, it was applied on continuous exponential utility functional employing approximate algorithmwhere the certainty is found to be, $c = \frac{\left(\mu_Y \alpha^2 + \alpha\right) \pm \sqrt{1+\alpha^2}}{\alpha}$. However, the minimum premium which the insurance manager will charge has been proved by the use of Taylor's expansion to be $\Sigma^- = \int_0^\infty y f_Y(y) dy$ which interestingly is the expected value of risk Y.

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